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STUDY OF  
NONSTATIONARY RANDOM PROCESS THEORY

By:

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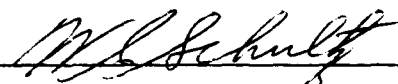
PROJECT ETUDE III

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## ABSTRACT

This report deals with methods for analyzing nonstationary processes in nonreal-time (computer-time) applications. In many cases immediate read-out of a nonstationary statistical analysis is not required, and therefore more accurate off-line analysis may be performed. A theory for nonreal-time correlation analysis and a theory for nonreal-time spectrum analysis are presented. These theories do not require that the assumptions of stationarity and ergodicity be made or even approximated. Instead, the theories are based on approximation of the expectation definition of the correlation function or its Fourier transform. Accordingly, correlation functions and spectra containing running-time axes may be postulated legitimately.

The theories make use of test functions for optimization of the analyzer configurations. Test functions are used in the optimization processes to avoid the need for precise a priori knowledge of the nonstationary correlation function or spectrum being estimated.

Finally, an experimental verification of the correlation theory is presented. Digital programs and plotting routines were used to obtain nonstationary correlation function estimates for data with known correlation functions, thereby making possible an analysis of estimation errors. In addition, a nonstationary correlation function estimate was obtained for flexible booster vibration data. This analysis shows the feasibility of the nonstationary theory.

## I. INTRODUCTION

Of great importance in the design of aerospace equipment is the problem of statistically analyzing signals whose sources of generation have time-varying parameters. Nonstationary signals are prevalent in aerospace engineering because of the varying environment through which a launch vehicle must travel, varying parameters within the vehicle, or because tests must be transient in nature. Moreover, communications and telemetering systems are subject to time-varying medium disturbances, the result of which is nonstationary signal reception.

In addition to aerospace engineering, nonstationary signals are prevalent in other branches of science and technology. Doppler weather radar signals, speech waveforms, and seismic waveforms, are important examples. Thus, nonstationary analysis methods developed for aerospace application are also widely applicable in other branches of science and technology.

Because of the prevalence of nonstationary signals, it is important that methods be developed for analyzing and understanding them. Under support of the National Aeronautics and Space Administration\*, a theory real-time correlation analysis of nonstationary signals was evolved,<sup>2,3</sup> which placed the analysis of nonstationary signals on a firm mathematical foundation. The real-time correlation theory (in a discrete version) was subsequently implemented on the digital computer.

During the past year under MSFC contract NAS8-11346, nonstationary signal analysis was carried further so that maximum advantage could be taken of the capabilities of the high-speed digital computer. The real-time correlation theory was extended to cover the nonreal-time or off-line (computer-time) case. This correlation theory, which is optimal in a continuous, nonreal-time sense, is fully described in Chapter II of this report.

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Because of the complexity of the nonreal-time nonstationary correlation theory, an experimental study which verifies the theory was performed. This experimental study involved four aspects: 1) computation of numerical examples of optimal filters for the correlation theory, 2) development of running digital computer programs and plotting subroutines which implement the nonreal-time nonstationary correlation theory, 3) experimental testing of the correlation theory using known test functions, and 4) computation of the nonstationary correlation function (and nonstationary spectrum by a suitable transform) for flexible booster data furnished by MSFC. The results of the experimental study are described in Chapter III of this report.

While the nonreal-time nonstationary correlation theory was being developed, a parallel effort was made under NAS8-11346 to develop a theory of direct spectrum analysis of nonstationary signals. Here, the idea was to obtain optimal estimates of nonstationary spectral densities, rather than transform optimal estimates of nonstationary correlation functions. This nonstationary spectral theory is described in Chapter IV. The reader is cautioned that the results of Chapter IV are preliminary, since limited scope of effort precluded an experimental study of the nonstationary spectral theory.

Finally, in Chapter V of this report recommendations are made for further work on the nonstationary theories and their application to MSFC Computation Laboratory problems.

## II. NONREAL-TIME (OFF-LINE) CORRELATION ANALYSIS OF NONSTATIONARY SIGNALS

### A. BACKGROUND

For many years engineers and scientists have found it advantageous to characterize randomly fluctuating phenomena by means of a single highly descriptive function. The correlation function and the power spectral density are most often used, since each contains a great deal of information. These functions provide the vital link between the raw data of an experiment on the one hand and the design or redesign of a system on the other. In addition, they are the required information for optimal linear least squares separation of signals from unwanted noise, and thus are important to communications engineering.

Correlation functions and power spectral densities are rather elusive when being measured. Extreme care must be taken to insure that the measurement method itself does not introduce artifacts and large errors. In addition, fundamental and economic limitations on data gathering make it necessary to deal with data records of finite length and number, resulting in additional sources of error. Consequently, the analysis of acoustical, vibrational, or electrical signals, requires great care to achieve full limiting accuracy for the data available. The problem of correlation and spectral accuracy is well recognized, and has been the subject of intensive study by communications engineers and other investigators for many years.<sup>1</sup>

Although the problem associated with accuracy has been carefully examined, there is a second type of problem that has not received the same degree of attention. This second problem results from the assumption made in conventional correlation and spectrum analysis that the data or waveforms being analyzed are generated by a stationary process. A stationary process is one in which any probability statement about the waveform values at specified times remains true if all the times are uniformly shifted by any given constant amount. Physical interpretation of this definition implies that a stationary process is one in which the underlying random waveform generating mechanism does not change with time.

Unfortunately, very few processes can be totally justified as stationary, and therefore it has often been necessary to resort to approximation of the stationary case in some way if a correlation or spectral analysis is to be performed. The assumption that a signal is generated by a stationary process is restrictive. Conventional stationary estimation theory does not apply adequately to many important waveforms and random signals in which the parameters generating the process do vary with time, or in other words, are nonstationary.

This section of the report presents and justifies a method of off-line correlation function estimation for signals generated by a nonstationary process. The theory underlying the method is postulated in a way which makes the assumption of stationarity unnecessary. Although errors in the correlation function estimation procedure will result, these errors are minimized according to a test function criterion.

A recent technical paper discussed a theory and method for correlating, in real-time, signals that are generated by nonstationary processes.<sup>2</sup> The theory was developed for on-line, physically realizable analysis and is limited to one-dimensional filtering operations. Thus, the method can be applied using standard analog or hybrid computer techniques. Except for the pure delay in the correlator, no data storage is required. In this section of the report an off-line theory is presented which is a modification and extension of the real-time theory of reference [2].

There are many practical nonstationary data processing problems where immediate read-out of the correlation function is not required. In these problems a period of time may elapse between generation of the signals and computation of the correlation function. More data can then be made available for processing because, at any given point in running time, both past and future data may be used for the correlation analysis. As a result of the greater amount of data, the errors involved in nonstationary correlation analysis may be made smaller in the nonreal-time (off-line) case.

Other advantages are associated with the nonreal-time approach to correlation analysis. First, correlation functions may be computed for both positive and negative shifts between signal pairs; that is, the correlation function may be computed as a two-sided function of the delay variable  $\tau$ . Also, since realizability conditions need not be specified for the filters in the correlator, rather general correlator derivations can be handled without serious complication. Finally, and perhaps the greatest advantage of all, the high-speed digital computer can be used for computation of nonreal-time correlation functions.

Although it is possible to develop more general and more sophisticated approaches than that presented here, it is probable that these would not be practical because of excess computation time or storage. The nonreal-time approach presented here limits the operations on the data to one-dimensional operations; that is, filtering operations contain only one independent variable. By so-limiting the approach, spatial filtering of large data arrays is eliminated. Storage and computation times can thereby generally be brought within practical bounds. The philosophy of approach used herein is similar to that of references [2] and [3]. Other philosophies can be developed, some of which are presented in references [4] through [13].

## B. DEFINITIONS AND PROBLEM STATEMENT

To estimate the correlation function of a nonstationary process it is first necessary to review the definition of the correlation function and to show that certain concepts associated with stationary processes may not be used when dealing with nonstationary processes. Let  $\{i_1(t); i_2(t)\}; n=1, 2, \dots, P$  represent a sequence of pairs of real signals that have been generated by the same nonstationary random process. The independent variable  $t$  usually represents running time, but can also represent a distance measure in some cases. This sequence represents the given information about the process from which the correlation function is to be estimated. If  $P=1$  then a single pair of waveforms is available; if  $P=2$ , then two pairs are available, and so on. If autocorrelation analysis is to be performed, the theory can be applied by letting  $i_2(t) = i_1(t)$  for each value of  $n$  up to  $P$ .



The crosscorrelation function, whether the process is stationary or nonstationary, is defined as the expected value of the lagged product of  ${}_n i_1(t)$  and  ${}_n i_2(t)$ :

$$\phi_{12}(t, \tau) = E[{}_n i_1(t) {}_n i_2(t - \tau)] \quad (1)$$

where  $n$  is an arbitrary integer and  $\tau$  represents a time translation between the two signals.\* Computation of this expectation for the general case will require knowledge of the joint probability density function of the signals  ${}_n i_1(t)$  and  ${}_n i_2(t - \tau)$ . However, if the process may be assumed stationary, then  $\phi_{12}(t, \tau)$  remains invariant regardless of the value of  $t$  and may therefore be written as a function of  $\tau$  only. Further, if the process possesses the additional property of ergodicity, then the correlation function may be computed from a suitably long time average of the lagged product  ${}_n i_1(t)$ ,  ${}_n i_2(t - \tau)$  without requiring the use of the other member pairs of the ensemble. Conventional correlation and spectral estimation is based upon the assumptions of stationarity and ergodicity, so that only one pair of data waveforms need be obtained for analysis.

If the process is nonstationary, the estimation procedure cannot be based on the theory associated with stationary and ergodic processes. Instead, the expectation definition of the correlation function must be considered as the starting point. Beginning with the expected value definition given above it is possible to show, using the law of large numbers, that in most cases  $\phi_{12}(t, \tau)$  would be equal to the limiting sample mean of lagged signal product pairs. Thus,

$$\phi_{12}(t, \tau) = \lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{{}_n i_1(t) {}_n i_2(t - \tau)}{N} \quad (2)$$

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\*In this definition the mean values of  ${}_n i_1(t)$  and  ${}_n i_2(t)$  are not removed. For a zero mean process, this definition corresponds to the conventional definition.

when the right hand side of this equation exists. Note that  $\phi_{12}(t, \tau)$  in this equation as well as in the expectation definition equation possesses two independent variables  $t$  and  $\tau$ . This correlation function is therefore capable of exhibiting changes as a function of time,  $t$ . In contrast, a correlation function equation based on stationary and ergodic assumptions precludes the capability of exhibiting changes in time, because the process is postulated as invariant under time translations.

Equation 2 cannot be used directly for correlation analysis of nonstationary signals, because it requires access to the total ensemble of signal pairs. However,  $\phi_{12}(t, \tau)$  may be considered as an ideal correlation function that is to be approximated by operating on the given signal pair sequence  $\{n i_1(t); n i_2(t)\}; n = 1, 2, \dots, P$ . This problem of approximating  $\phi_{12}(t, \tau)$  with least error, by processing the given  $P$  pairs of signals is considered in this report as the fundamental objective of non-stationary correlation analysis.

A solution to this problem may be obtained if it is carefully specified and limited. Define  $\theta_{12}(t, \tau)$  as the sample mean of the given signal lagged products:

$$\theta_{12}(t, \tau) = \sum_{n=1}^P \frac{n i_1(t) n i_2(t - \tau)}{P} \quad (3)$$

where again  $\tau$  represents the delay or displacement variable. Further, define a noise-like difference as

$$\eta_{12}(t, \tau) = \theta_{12}(t, \tau) - \phi_{12}(t, \tau) \quad (4)$$

where  $\eta_{12}(t, \tau)$  is assumed extraneous to the correlation function  $\phi_{12}(t, \tau)$ . In other words,  $\theta_{12}(t, \tau)$  is composed of two components: a desired component  $\phi_{12}(t, \tau)$  and an undesired component  $\eta_{12}(t, \tau)$ . The objective will be to operate on the computable function  $\theta_{12}(t, \tau)$  in a way which suppresses the  $\eta_{12}(t, \tau)$  component and causes least distortion of the  $\phi_{12}(t, \tau)$  component.

It is necessary to specify a class of admissible operations on the function  $\theta_{12}(t, \tau)$  that may be used in approximating  $\phi_{12}(t, \tau)$ . The class should be chosen so that accurate approximation is possible, yet it should not be chosen so general as to make the computations difficult. Let the output of the correlation operation (or analyzer) be defined as  $\psi_{12}(t, \tau)$ . Then choose the class of operations such that they may be described as two one-dimensional filtering operations on  $\theta_{12}(t, \tau)$ ; that is, let\*

$$\psi_{12}(t, \tau) = \iint h(\alpha) l(\gamma) \theta_{12}(t - \alpha, \gamma + \tau) d\gamma d\alpha \quad (5)$$

where  $h(\alpha)$  and  $l(\gamma)$  are impulse response functions that are to be specified by the subsequent optimization process. The functions  $h(\alpha)$  and  $l(\gamma)$  are to be chosen in a way that minimizes some measure of the difference between  $\phi_{12}(t, \tau)$  and  $\psi_{12}(t, \tau)$ .

Equation 5 describing the class of admissible operations on  $\theta_{12}(t, \tau)$  has not been chosen completely arbitrarily. Advantages are associated with this particular choice of a class. These will be discussed at the end of section II, wherein it will be shown that alternate methods of computation can be developed for performing operations described by equation 5.

It is legitimate to ask why a filtering operation on the sample mean of the given signal lagged products should improve the estimate of the correlation function. One might question the validity of performing any type of filtering on  $\theta_{12}(t, \tau)$ . There is a twofold reason through which improvement can be obtained by filtering. On the one hand, the difference function  $n_{12}(t, \tau)$  is unwanted, and its amplitude can be reduced by the proper form of filtering. On the other hand, ensemble averaging will ordinarily have a tendency to smooth, and thus  $\phi_{12}(t, \tau)$  is a smoother function than  $\theta_{12}(t, \tau)$ . Filtering is capable of introducing smoothing in a way that approximates ensemble averaging.

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\*When limits on integrals are deleted, they are to be taken as  $\infty$  at the upper limits and  $-\infty$  at the lower limits.

### C. DEVELOPMENT OF A PERFORMANCE MEASURE

As discussed in references [2] and [3], three sources of error should be taken into account in the correlation analysis of nonstationary signals. The problem at this point becomes one of mathematical specification of these three sources of error in terms of  $h(\alpha)$  and  $l(\gamma)$  and the subsequent selection of  $h(\alpha)$  and  $l(\gamma)$  such that the errors are minimized. Substitution of (4) into (5) yields

$$\begin{aligned} \psi_{12}(t, \tau) = & \iint h(\alpha) l(\gamma) \phi_{12}(t - \alpha, \gamma + \tau) d\gamma d\alpha \\ & + \iint h(\alpha) l(\gamma) \eta_{12}(t - \alpha, \gamma + \tau) d\gamma d\alpha \end{aligned} \quad (6)$$

which makes possible the examination of the sources of error. The first term on the right shows that the filtering operation will produce distortion of the correlation function. Distortion will exist as a function of  $\tau$ , because of the filtering operation represented by  $l(\gamma)$ . Distortion will also exist as a function of  $t$  because of the operation represented by  $h(\alpha)$ . In addition, examination of the second term on the right shows that the entire quantity must be considered as error, since it represents the filtering of an extraneous or unwanted component of the function  $\theta_{12}(t, \tau)$ . Thus, the three sources of error may be classified arbitrarily as:

1. distortion of the correlation function as a function of  $t$ ,
2. distortion of the correlation function as a function of  $\tau$ , and
3. noise or instability resulting from the extraneous component.

#### Measures of Distortion in $t$ and $\tau$

In order that the distortion of the correlation function in  $t$  may be minimized, a test-type correlation function is chosen. This test function should be representative (in the  $t$  dimension) of those being detected. It is postulated that precise knowledge of the correlation function is unavailable, since that type of knowledge would preclude the need for performing a cor-

relation analysis. Test-function testing of the correlation operation (or analyzer) affords a good, workable compromise between total absence and total presence of knowledge regarding the process.

In his review of the real-time nonstationary correlation theory of reference [2] Larrowe pointed out the disadvantage that the test functions must be selected before the optimization procedure may be performed.<sup>14</sup> In contrast, this nonreal-time theory does not require selection of specific test-functions in the derivation, because absence of physical realizability conditions simplifies the analysis somewhat. In this theory a general form may be carried through the analysis, allowing the test function to be chosen in each particular application. Thus, for testing distortion in  $t$  let

$$\phi_{12}(t, \tau) = \lambda_t q(t) \quad \text{for all } \tau \quad (7)$$

where  $q(t)$  is the test function for the  $t$ -axis, and  $\lambda_t$  is an arbitrary positive constant determining the weighting of the test function. The function  $q(t)$  is made uniform in  $\tau$  in order that separation of distortion in  $t$  and distortion in  $\tau$  may be accomplished.

It is important to assure that biases in the input data are correctly reflected in the output of the correlator. This condition can be attained approximately if the class of test functions is limited to those that have the property

$$q(t) = \int_{-\infty}^t q_1(\sigma) d\sigma \quad (8)$$

where  $q_1(\sigma)$  is such that  $q_1(0) \neq 0$ .<sup>\*</sup> Equation 8 may be written in the form:

$$q(t) = \int q_1(\sigma) u_1(t - \sigma) d\sigma \quad (9)$$

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<sup>\*</sup>In this report, functions of  $S$  written with upper-case letters are Fourier transforms of their corresponding lower-case counterparts. For example,  $Q_1(S) = \int q_1(\sigma) e^{-jS\sigma} d\sigma$ .

where  $u_-(t)$  is the unit step function. Then the Fourier transform of (9) yields

$$Q(s) = \frac{Q_-(s)}{s} \quad (10)$$

an equation that will be used later.

A measure of the distortion in  $t$  for the chosen test correlation function is easily obtained by squaring and integrating the difference between the output of correlation analyzer and the test function itself. Thus, let the measure of distortion be defined as

$$P_t = \lambda_t^2 \int \left[ q(t) - \iint h(\alpha) l(\gamma) q(t-\alpha) d\gamma d\alpha \right]^2 dt \quad (11)$$

which will serve to assess the first source of error in nonstationary correlation analysis.

Distortion in  $\tau$  may be handled in an analogous manner. For testing, let

$$\phi_{t\tau}(t, \tau) = \lambda_\tau r(\tau) \quad \text{for all } t, \quad (12)$$

where  $r(\tau)$  is the test function and  $\lambda_\tau$  is another arbitrary positive weighting constant. The test function is made uniform in  $t$  (which is equivalent to being stationary), in order that separation of this source of distortion may be attained.

Again, the test function class is further restricted to insure correct bias indication by the correlation analyzer. Let

$$r(\tau) = \int_{-\infty}^{\tau} r_1(\sigma) d\sigma \quad (13)$$

where  $R_1(s)$  is such that  $R_1(0) \neq 0$ . Then,

$$R(s) = \frac{R_1(s)}{s} \quad (14)$$

The distortion measure in  $\tau$  is then defined similarly as

$$P_{\tau} = \lambda_{\tau}^2 \int \left[ r(\tau) - \iint h(\alpha) l(\gamma) r(\gamma + \tau) d\gamma d\alpha \right]^2 d\tau \quad (15)$$

which makes possible the quantitative assessment of the second source of error.

#### Error Measure for the Extraneous Component

As stated earlier, an extraneous or noise-like component will be present in the output of the correlation analyzer, which is represented mathematically by the second term on the right side of (6). This term must be made as small as possible, because its entire contribution is extraneous by definition of  $n_{12}(t, \tau)$  in (4). To properly assess this third source of error, a test function and performance measure must be selected.

Consider that the function  $\phi_{12}(t, \tau)$  and the function  $\theta_{12}(t, \tau)$  can be visualized as two-dimensional surfaces or arrays. The amount by which they differ is again a two-dimensional surface,  $n_{12}(t, \tau)$ . Thus,  $n_{12}(t, \tau)$  is a two-dimensional noise waveform. Suppose that the noise test function is chosen so as to be of infinite length in both  $t$  and  $\tau$  and stationary in both  $t$  and  $\tau$ . Further assume that an average over  $t$  and an average over  $\tau$  of the lagged product yields

$$\overline{\overline{n_{12}(t, \tau) n_{12}(t - \alpha, \tau + \gamma)}}} = a_n(\alpha) b_n(\gamma) \quad (16)$$

where the double bar indicates  $t$  and  $\tau$  averaging. Note that  $a_n(\alpha)$  and  $b_n(\gamma)$  must always be even functions if  $n_{12}(t, \tau)$  is stationary and ergodic in both  $t$  and  $\tau$ . Let the noise test function be specified as unbiased as a function of both  $t$  and  $\tau$ , so that a solution to the problem may be obtained which allows the correlator to correctly indicate biases in the data.\*

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\*Correct readout of the biases is considered a desirable property of the correlator. Therefore, the noise component test function should not contain a bias, since this would cause minimization of a desired function.

If, for the noise test function described above, the correlation analyzer output is squared and averaged over  $t$  and  $\tau$ , a total measure of the instability or extraneous noise response will be obtained. Let

$$P_n \equiv \left[ \iint h(\alpha) l(\gamma) n_{12}(t-\alpha, \gamma+\tau) d\gamma d\alpha \right]^2 \quad (17)$$

$$= \iiint h(\alpha_1) h(\alpha_2) l(\gamma_1) l(\gamma_2) \overline{n_{12}(t-\alpha_1, \gamma_1+\tau) n_{12}(t-\alpha_2, \gamma_2+\tau)} d\gamma_1 d\gamma_2 d\alpha_1 d\alpha_2 \quad (18)$$

$$= \iint h(\alpha_1) h(\alpha_2) a_n(\alpha_2 - \alpha_1) d\alpha_1 d\alpha_2 \cdot \iint l(\gamma_1) l(\gamma_2) b_n(\gamma_2 - \gamma_1) d\gamma_1 d\gamma_2 \quad (19)$$

Equation 19 describes the noise response or instability in a simple measure.\* The functions  $a_n(\alpha)$  and  $b_n(\gamma)$  can be considered as time-averaging correlation functions of the extraneous component  $n_{12}(t, \tau)$  along each of its two independent variables. An investigator may choose these two correlation functions to be representative of  $n_{12}(t, \tau)$  in each specific case.

#### The Total Performance Measure

A total assessment of the errors involved in nonstationary non-real-time correlation analysis can be obtained by summing the performance measures representing the three sources of error. Let the total performance measure be defined as:

$$\Theta = P_t + P_\tau + P_n \quad (20)$$

By changing the values of the arbitrary parameters  $\lambda_t$  and  $\lambda_\tau$ , the various sources of error may be traded against one another.

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\*Note that  $P_n$  does not contain an arbitrary positive weighting constant as do  $P_t$  and  $P_\tau$ . The solution can always be normalized so that any constant association with  $P_n$  may be removed. Therefore, such a constant is superfluous.



The point has been made that the correlation analyzer must indicate at its output the correct value of the biases in the input data. One way of insuring approximately correct bias reading is to constrain the zero frequency gains of the two filtering operations represented by  $h(\alpha)$  and  $l(\gamma)$  to be equal to unity. Therefore, the following two constraint conditions are to be incorporated in the solution for minimum correlator error:

$$\int h(\alpha) d\alpha - 1 = 0 \quad (21)$$

$$\int l(\gamma) d\gamma - 1 = 0 \quad (22)$$

The problem then becomes one of minimizing  $\Theta$  subject to satisfaction of the constraints specified by (21) and (22).

It should be realized that the development of a performance measure is to a great extent arbitrary. Many different approaches might have been taken. The one presented here has been selected because it is relatively general, it leads to a precise optimal solution (in the sense of the chosen performance measure), and it correctly reflects the sources of error and desired constraints. Of major importance here are the facts that this approach exhibits the fundamental errors of nonstationary correlation analysis and that the errors are minimized to the extent possible.

#### D. CORRELATION ANALYZER DETERMINATION

The minimization of the performance measure is accomplished by determining the extremals of the performance measure with Lagrange multiplier side conditions. Two variational equations will be obtained because both  $h(\alpha)$  and  $l(\gamma)$  must be determined. These equations are

$$I_{\ell} \equiv \lambda_t^2 \int \left\{ q(t) - \int [l(\gamma) + c\eta(\gamma)] d\gamma \int h(\alpha) q(t-\alpha) d\alpha \right\}^2 dt \\ + \lambda_t^2 \int \left\{ r(t) - \int h(\alpha) d(\alpha) \int [l(\gamma) + c\eta(\gamma)] r(\gamma+\tau) d\gamma \right\}^2 d\tau \quad (23)$$

(equation continued on next page)

$$\begin{aligned}
& + \iint h(\alpha_1) h(\alpha_2) a_n(\alpha_2 - \alpha_1) d\alpha_2 d\alpha_1 \cdot \iint [\ell(\gamma_1) + \epsilon \eta(\gamma_1)] [\ell(\gamma_2) + \epsilon \eta(\gamma_2)] \\
& \quad \times b_n(\gamma_2 - \gamma_1) d\gamma_2 d\gamma_1 \quad (23) \\
& + \lambda_1 \left\{ \int h(\alpha) d\alpha - 1 \right\} + \lambda_2 \left\{ \int [\ell(\gamma) + \epsilon \eta(\gamma)] d\gamma - 1 \right\} \quad \text{cont.}
\end{aligned}$$

and a similar equation with  $h(\alpha)$  subjected to a variation instead of  $\ell(\gamma)$ . In these equations,  $\lambda_1$  and  $\lambda_2$  are Lagrange multipliers to be determined in such a way that the constraint conditions (21) and (22) are satisfied.

The extremals of (23) are obtained by performing the following operation on  $I_\ell$ :

$$\left. \frac{\partial I_\ell}{\partial \epsilon} \right|_{\epsilon=0} = 0 \quad (24)$$

Evaluating (24) and interchanging orders of integration\* yields

$$\begin{aligned}
& \int \left\{ -2\lambda_1^2 \int [q(t) - \int \ell(\gamma_1) d\gamma_1 \cdot h_1(t)] h_1(t) dt \right. \\
& \quad - 2\lambda_2^2 \int [r(t) - H(0) \int \ell(\gamma_1) r(\tau + \gamma_1) d\gamma_1] H(0) r(\tau + \gamma) d\tau \\
& \quad \left. + 2H_2 \int \ell(\gamma_1) b_n(\gamma - \gamma_1) d\gamma_1 + \lambda_2 \right\} \eta(\gamma) d\gamma = 0 \quad (25)
\end{aligned}$$

where

$$H(0) = H(S) \Big|_{S=0} = \int h(\alpha) d\alpha \quad (26)$$

$$h_1(t) \equiv \int h(\alpha) q(t - \alpha) d\alpha \quad (27)$$

$$H_2 \equiv \iint h(\alpha_1) h(\alpha_2) a_n(\alpha_2 - \alpha_1) d\alpha_2 d\alpha_1 \quad (28)$$

---

\*Interchange of orders of integration, when performed as indicated in (25), can be easily justified for most sets of practical test functions.

Application of the fundamental theorem of the calculus of variations allows the extremal condition to be written as:

$$\begin{aligned}
& -\lambda_t^2 \int \left[ q(t) - \int \ell(\gamma_i) d\gamma_i \cdot h_i(t) \right] h_i(t) dt \\
& -\lambda_\tau^2 \int \left[ r(\tau) - H(0) \int \ell(\gamma_i) r(\tau + \gamma_i) d\gamma_i \right] H(0) r(\tau + \gamma) d\tau \\
& + H_2 \int \ell(\gamma_i) b_n(\gamma - \gamma_i) d\gamma_i + \frac{1}{2} \lambda_2 = 0
\end{aligned} \tag{29}$$

The second and third terms of this equation are functions of  $\gamma$ , whereas the first and fourth terms are not functions of  $\gamma$ . Suppose  $\lambda_2$  is chosen such that

$$\frac{1}{2} \lambda_2 = \lambda_t^2 \int \left[ q(t) - \int \ell(\gamma_i) d\gamma_i \cdot h_i(t) \right] h_i(t) dt \tag{30}$$

Then, if the resulting extremal equation satisfies the constraint condition, the equation is a solution to the problem. Substituting (30) into (29), the extremal equation becomes

$$\begin{aligned}
& -\lambda_\tau^2 \int \left[ r(\tau) - H(0) \int \ell(\gamma_i) r(\tau + \gamma_i) d\gamma_i \right] H(0) r(\tau + \gamma) d\tau \\
& + H_2 \int \ell(\gamma_i) b_n(\gamma - \gamma_i) d\gamma_i = 0
\end{aligned} \tag{31}$$

This integral equation may be solved by Fourier transforming the individual terms. The resulting equation is (making use of (14)):

$$-\lambda_\tau^2 H(0) \frac{R_1(s)}{s} \left[ \frac{R_1(-s)}{-s} - H(0) \frac{R_1(-s)}{-s} L(s) \right] + H_2 B_n(s) L(s) = 0 \tag{32}$$

Rearrangement yields

$$L(s) = \frac{\lambda_\tau^2 H(0) \frac{R_1(s)}{s} \frac{R_1(-s)}{-s}}{H_2 B_n(s) + \lambda_\tau^2 H^2(0) \frac{R_1(s)}{s} \frac{R_1(-s)}{-s}} \tag{33}$$

In order that a check might be made to insure that the constraint condition is satisfied, the limit as  $S \rightarrow 0$  of (32) is taken. The result is

$$L(0)H(0) = 1 \quad (34)$$

because for unbiased noise test functions  $\lim_{S \rightarrow 0} B_n(S)$  is finite. Thus, the constraint conditions can be satisfied, since (34) admits the conditions  $L(0) = 1$  and  $H(0) = 1$ . The final expression for  $L(S)$  becomes\*

$$L(S) = \frac{\left(\frac{\lambda_r^2}{H_2}\right) \frac{R_1(S)}{S} \frac{R_1(-S)}{-S}}{B_1(S) B_1(-S) + \left(\frac{\lambda_r^2}{H_2}\right) \frac{R_1(S)}{S} \frac{R_1(-S)}{-S}} \quad (35)$$

where, since  $B_n(S)$  is even in  $S$ , it may be written as:

$$B_n(S) \equiv B_1(S) B_1(-S) \quad (36)$$

The solution for  $h(\alpha)$  is obtained in a similar fashion, however, signs within the derivation are different. Briefly,

$$-\lambda_t^2 \int \left[ q(t) - L(0) \int h(\alpha_1) q(t - \alpha_1) d\alpha_1 \right] L(0) q(t - \alpha) dt + L_2 \int h(\alpha_1) a_n(\alpha - \alpha_1) d\alpha_1 = 0 \quad (37)$$

$$-\lambda_t^2 L(0) \left[ \frac{Q_1(S)}{S} - L(0) H(S) \frac{Q_1(S)}{S} \right] \frac{Q_1(-S)}{-S} + L_2 A_n(S) H(S) = 0 \quad (38)$$

$$H(S) = \frac{\left(\frac{\lambda_t^2}{L_2}\right) \frac{Q_1(S)}{S} \frac{Q_1(-S)}{-S}}{A_1(S) A_1(-S) + \left(\frac{\lambda_t^2}{L_2}\right) \frac{Q_1(S)}{S} \frac{Q_1(-S)}{-S}} \quad (39)$$

$$\text{where} \quad A_n(S) = A_1(S) A_1(-S) \quad (40)$$

---

\*Proof that  $L(S)$  as given in (35) produces minimum correlator error (according to the performance measure with constraints incorporated) may be obtained in a straightforward manner. The extremal condition (25) is substituted into the expression for  $I_L - \Theta$ . Then, the resulting expression is shown to be greater than or equal to zero for all  $\mathcal{C}$  and  $\eta(\gamma)$ .

Equations 35 and 39 specify that the transfer functions of the two optimum filters to be used in the correlation operation described in (5).

The theory, as described in this report, has been derived in a relatively general fashion, to provide flexibility in its application. First, the number of member signal pairs has not been specified, and thus, correlation of one pair or several pairs is possible. Also, since the test functions have not been specified, these functions may be chosen in accordance with the problem at hand. Finally, since the theory has been developed for crosscorrelation, autocorrelation follows as a special case without modification.

It will be shown that there are two different ways in which the correlation operation of (5) may be implemented. First, the operation may be implemented by direct filtering of the array represented by  $\theta_{12}(t, \tau)$ . The steps to the computation are the following:

1. Compute the two-dimensional array representing  $\theta_{12}(t, \tau)$  by adding and storing the individual lagged products,  
 $i_1(t), i_2(t-\tau), i_1(t), i_2(t-\tau), \dots, i_P(t), i_P(t-\tau)$ .
2. Filter the array in each dimension. First, for each setting of  $t$ , filter the  $\tau$  dimension with a filter whose impulse response is  $l(\gamma)$ . Then in the resulting two-dimensional array, filter the  $t$  dimension for each setting of  $\tau$  with a filter whose impulse response is  $h(\alpha)$ . The resulting array is the output of the correlation operation.

The second way in which the operation may be performed is a result of the network properties of equation 5. Figure 1 shows a network configuration whose output for inputs  $i_1(t)$  and  $i_2(t)$  is:

$$\psi_{12}(t, \tau) = \iint h(\alpha) l(\gamma) i_1(t-\alpha) i_2(t-\alpha-\gamma-\tau) d\gamma d\alpha \quad (41)$$

Upon performing an average over the  $P$  pairs of signals, the resulting equation is (5). Thus, the network configuration may be used to compute

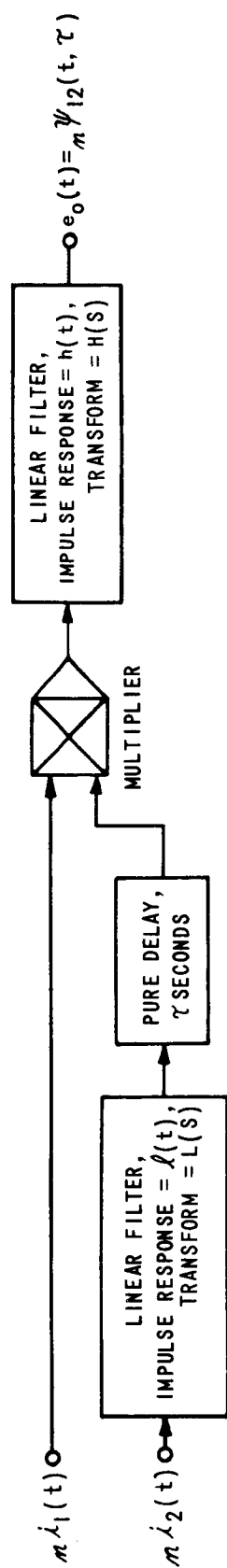


Figure 1 NETWORK CONFIGURATION THAT MAY BE USED FOR IMPLEMENTING THE NONREAL-TIME NONSTATIONARY CORRELATION THEORY.

each  $\pi \psi_{12}(t, \tau)$ . Averaging over the  $\rho$  output arrays then produces the correlation operation specified by equation 5.

The first approach requires more storage, but is probably more efficient. It is generally suitable for modern high-speed digital computers. The second approach allows the computation of correlation functions for a fixed  $\tau$  without operating on neighboring values of  $\tau$ . Thus, one line of  $t$  (with  $\tau$  fixed) may be processed and printed out before moving on to the next fixed value of  $\tau$ . The advantage of this second approach is that less storage is required. It is probably more suitable for smaller digital or hybrid computers.

### III. EXPERIMENTAL STUDY OF NONREAL-TIME CORRELATION ANALYSIS

A digital computer program was developed for performing nonreal-time nonstationary correlation analysis. The purpose of this program was twofold: first, it was to be used for experimental verification of the foregoing theory, and second, it was to be used to analyze a pilot run of NASA flexible booster data. Isometric plotting subroutines were also developed in order that the computer output might be displayed in graphical form. This section describes the results of the experimental study.

#### A. ANALYZER TESTS

The first tests performed with the theory and program involved the autocorrelation of an uncorrelated, nearly stationary noise waveform and the crosscorrelation of two uncorrelated waveforms that were independent of each other. For this test the Fourier transforms of the two filters were chosen as optimum for  $R_f(s)$ ,  $B_f(s)$ ,  $Q_f(s)$  and  $A_f(s)$  each equal to unity. The filters themselves were then of the forms

$$L(s) = \frac{0.5}{1 + \frac{s}{2\pi \cdot 1000}} + \frac{0.5}{1 - \frac{s}{2\pi \cdot 1000}} \quad (42)$$

and

$$H(s) = \frac{0.5}{1 + \frac{s}{25}} + \frac{0.5}{1 - \frac{s}{25}} \quad (43)$$

Figure 2 shows the computer plot of the steady-state analyzer output for the autocorrelation case (and for positive  $\tau$  only). Whereas, the true correlation function would exhibit an impulse "ridge" for  $\tau = 0$  and zero value elsewhere, the correlator output only approximates these conditions. For large values of  $\tau$ , the correlator output fluctuates about zero, thus exhibiting the instability of nonstationary correlation analysis. For values of  $\tau$  near zero, positive correlation is exhibited; the correlation is spread about  $\tau = 0$  instead of being bunched at  $\tau = 0$ . Thus, some distortion of the true correlation function occurs along the  $\tau$  axis. Notwithstanding these



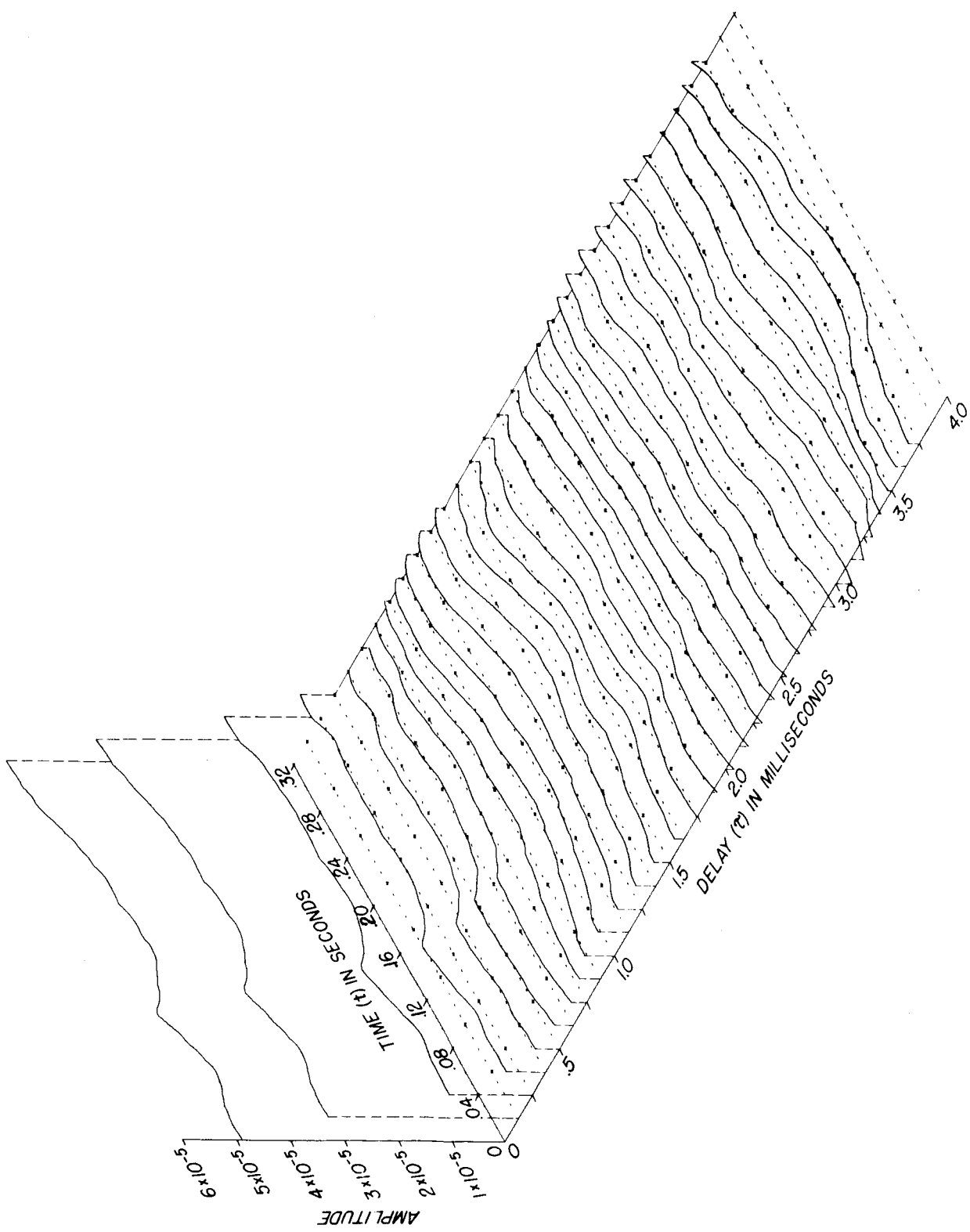


Figure 2 COMPUTER PLOT OF STEADY - STATE CORRELATION ANALYZER OUTPUT FOR AUTOCORRELATION OF UNCORRELATED NOISE (POSITIVE  $\tau$  ONLY).

shortcomings, it is rather clear that the correlator is operating as predicted in theory, and that the output is very valuable in characterizing the signal on which the correlator operated. It should be noted that the smoothing times of the two filters, represented by  $L(s)$  and  $H(s)$  were chosen to exhibit the sources of error involved in nonstationary analysis.

The crosscorrelation of uncorrelated noise produced a plot similar to that of Figure 2, except that there was no rise in output level for small values of  $\tau$ . In other words, the correlator output for every value of  $\tau$  fluctuated about zero. This result is as anticipated, since the true correlation function is zero everywhere.

In another test, two signals were crosscorrelated whose true crosscorrelation function was a stationary exponential:

$$\phi(t, \tau) = \begin{cases} k_0 e^{-200\tau} & ; \quad \tau \geq 0 \\ 0 & ; \quad \tau < 0 \end{cases} \quad (44)$$

For this test of the analyzer  $B_i(s)$ ,  $Q_i(s)$  and  $A_i(s)$  were again set equal to unity, while  $R_i(s)$  was made equal to the Fourier transform of (44):

$$R_i(s) = \frac{1}{1 + \frac{s}{200}} \quad (45)$$

The analyzer filters were then given by

$$L(s) = \frac{0.5 + 0.65 \times 10^{-3}s}{1 + 1.55 \times 10^{-3}s + 10^{-6}s^2} + \frac{0.5 - 0.65 \times 10^{-3}s}{1 - 1.55 \times 10^{-3}s + 10^{-6}s^2} \quad (46)$$

and

$$H(s) = \frac{0.5}{1 + \frac{s}{20}} + \frac{0.5}{1 - \frac{s}{20}} \quad (47)$$

Figure 3 is a computer plot of the steady-state correlator output for the exponential test (and for positive  $\tau$  only). Here, distortion of the true correlation function occurs as anticipated. In addition, the usual

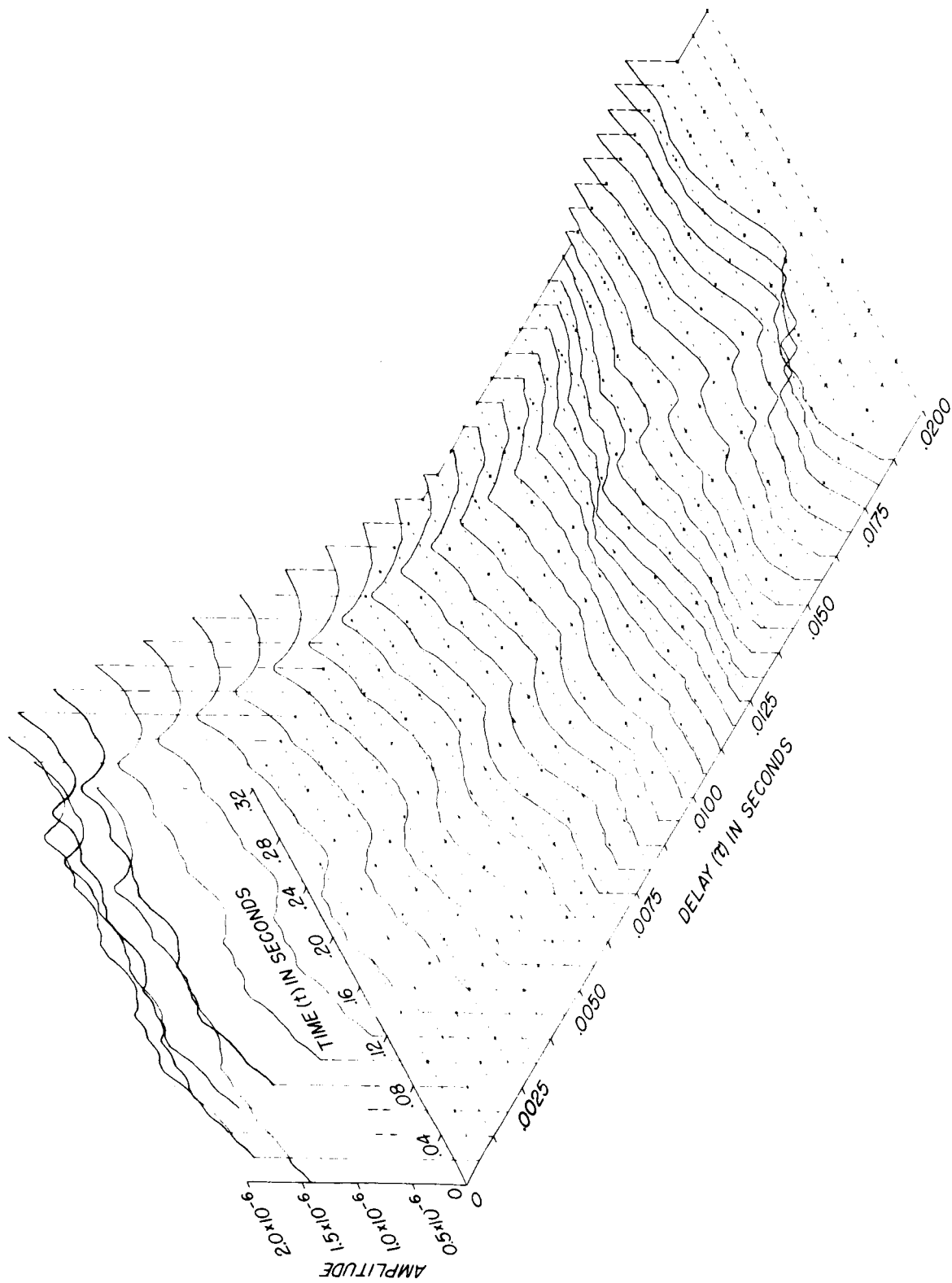


Figure 3 COMPUTER PLOT OF STEADY-STATE CORRELATION ANALYZER OUTPUT  
FOR CROSSCORRELATION OF SIGNALS WHOSE TRUE CROSSCORRELATION  
FUNCTION IS AN EXPONENTIAL (POSITIVE  $\tau$  ONLY).

instability exists as a result of finite smoothing time. It should be noted that because of a change in amplitude scale the instability appears greater here than in the uncorrelated noise test; actually the instability is smaller in the exponential test. Again, the information obtained from the correlator would be very useful in describing the relationship between the two input signals.

In the tests described thus far the smoothing represented by the filter  $H(S)$  was made finite even though the true correlation functions were nearly stationary. Of course, some instability results because of this finite smoothing. The advantage of using finite smoothing is that changes in time of the true correlation function may then be detected by the correlator. To exhibit this ability to detect changes, a test was performed in which the true correlation function undergoes an abrupt change in time. More specifically, two signals were generated whose true crosscorrelation function is uniform in  $\tau$  and undergoes a step change in  $t$  :

$$\phi_{12}(t, \tau) = \begin{cases} k_1 & ; \quad t < 0.16 \text{ sec.} \\ 2k_1 & ; \quad t \geq 0.16 \text{ sec.} \end{cases} \quad (48)$$

In this test  $R_1(S)$ ,  $B_1(S)$ ,  $Q_1(S)$  and  $A_1(S)$  were all set equal to unity once again. The filters used were

$$L(S) = \frac{0.5}{1 + \frac{S}{1000}} + \frac{0.5}{1 - \frac{S}{1000}} \quad (49)$$

and

$$H(S) = \frac{0.5}{1 + \frac{S}{20}} + \frac{0.5}{1 - \frac{S}{20}} \quad (50)$$

Figure 4 is a computer plot of the output of the correlator for the abrupt step in  $t$  test. It is seen that the correlator output smoothes the abrupt change, but does gradually assume the new level. And, it is clear that the uniform nature of the true correlation function along the  $\tau$  axis is being detected. Because of the symmetry of the filters in the correlator, it would be expected that the correlator's output at  $t = 0.16$  sec. should be

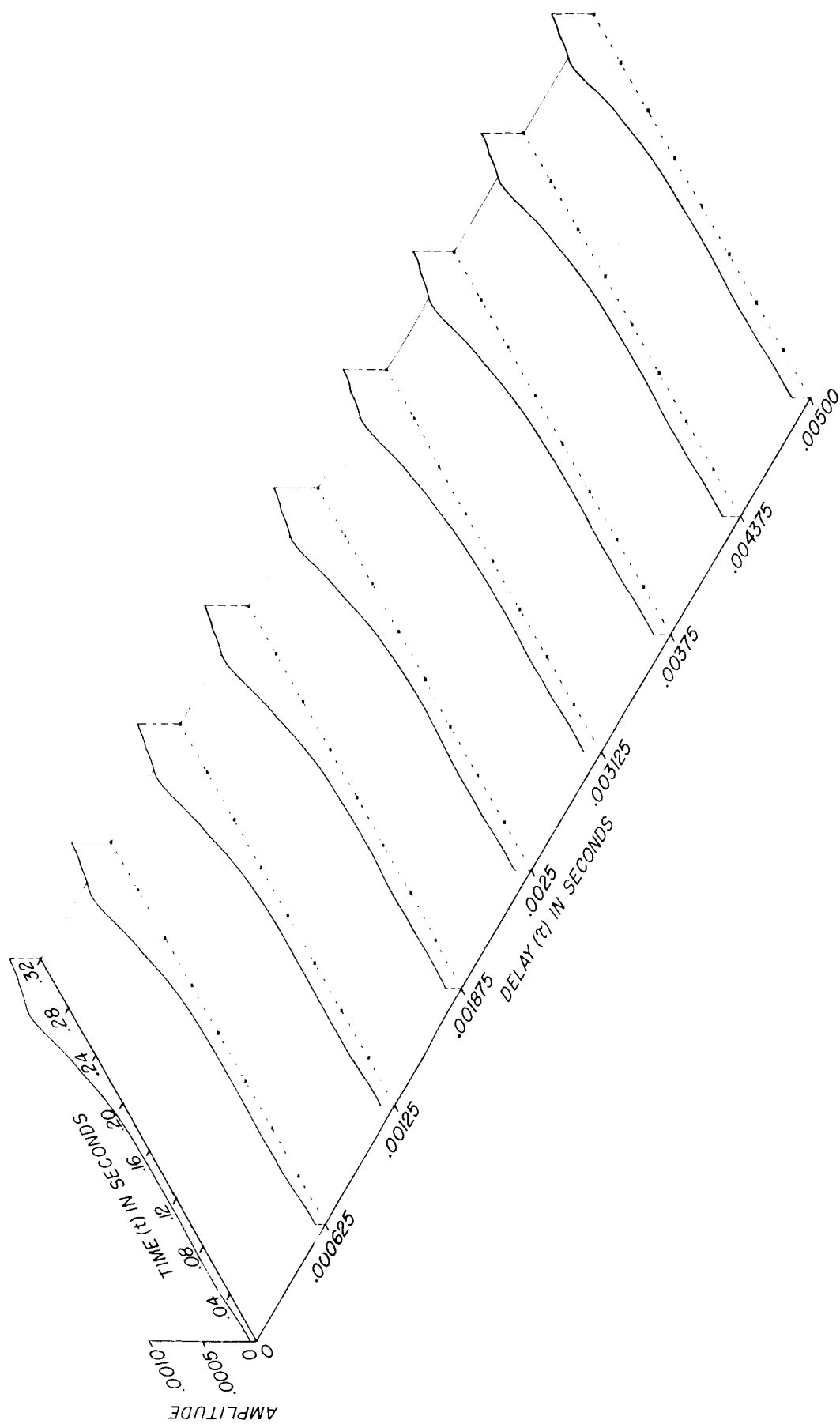


Figure 4 COMPUTER PLOT OF CORRELATION ANALYZER OUTPUT FOR CROSSCORRELATION OF SIGNALS WHOSE TRUE CROSSCORRELATION FUNCTION IS UNIFORM IN  $\tau$  AND UNDERGOES A STEP CHANGE IN  $t$  (POSITIVE  $\tau$  ONLY).

midway in its change from the old to the new value of the true correlation function. The plot shows that the correlator's output is late in rising to the new value; an explanation is that the original noise source is not precisely stationary or that the analyzer's instability causes errors in the output plot. Again, it becomes clear that the correlator's output would be valuable in characterizing the input signal pair. Additionally, the correlator is capable of following changes in the statistics of the input signals.

The tests described above give strong indication that the non-real-time correlation theory does perform as expected when implemented. Important features of each true correlation function are detected by the correlator, and the sources of error appear adequately taken into account.

#### B. APPLICATION TO BOOSTER DATA

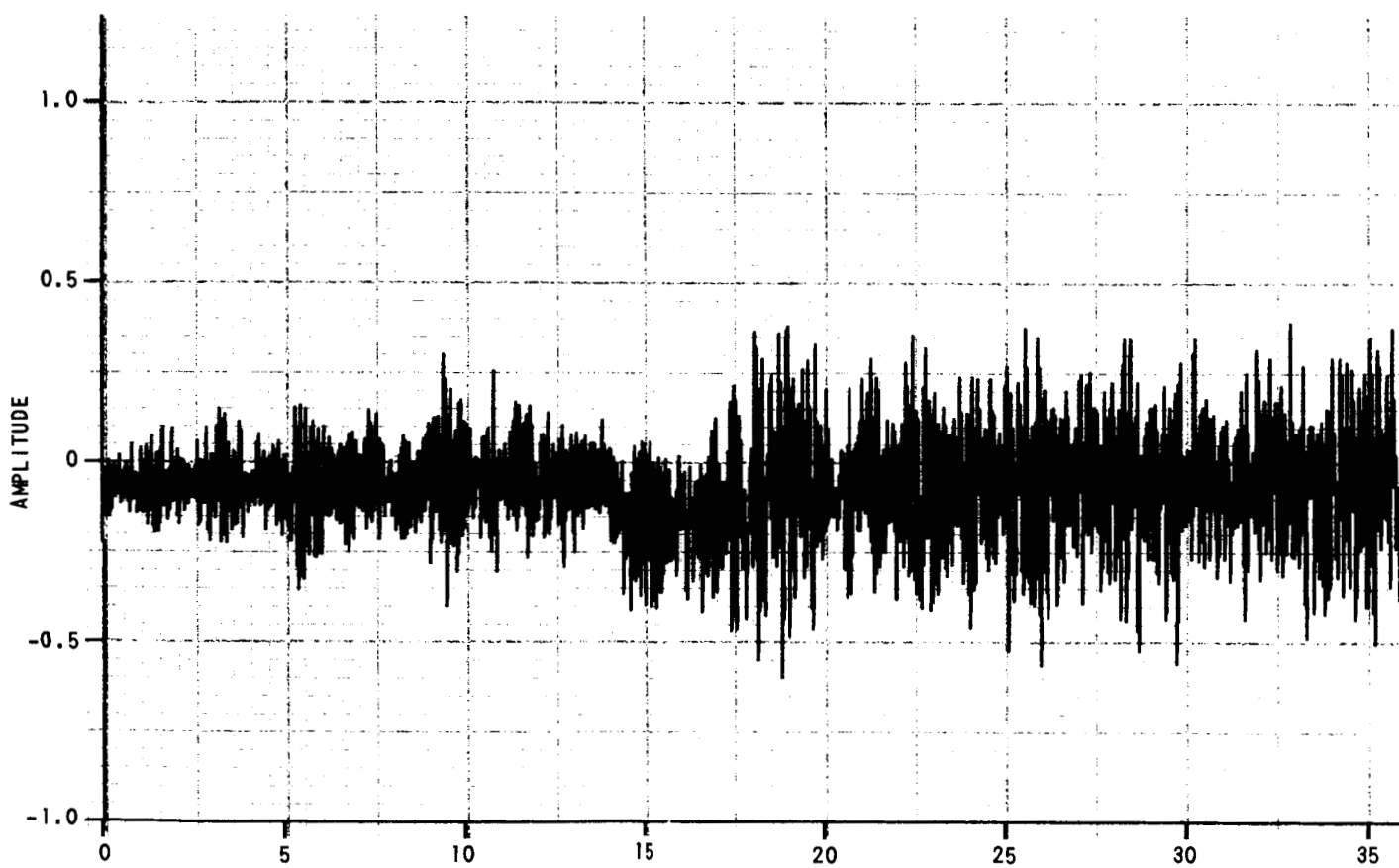
In the analyzer tests described above, all of the input signals were generated by the digital computer and were then used as inputs to the correlator program. The primary signal source was a computer algorithm for generation of approximately stationary and approximately uncorrelated signals. It seemed, therefore, that some type of test on actual experimental data ought to be performed. The Computation Laboratory of Marshall Space Flight Center, NASA, made available several records of flexible booster test vibration waveforms. From these, one was selected because of its apparent nonstationarity. Figure 5 is a plot of this waveform.

An autocorrelation analysis was performed using very simple test functions to optimize the analyzer;  $R_i(s)$ ,  $B_i(s)$ ,  $Q_i(s)$  and  $A_i(s)$  were all set equal to unity. The filters used were

$$L(s) = \frac{0.5}{1 + \frac{s}{100}} + \frac{0.5}{1 - \frac{s}{100}} \quad (51)$$

and

$$H(s) = \frac{0.5}{1 + 3s} + \frac{0.5}{1 - 3s} \quad (52)$$



28-1

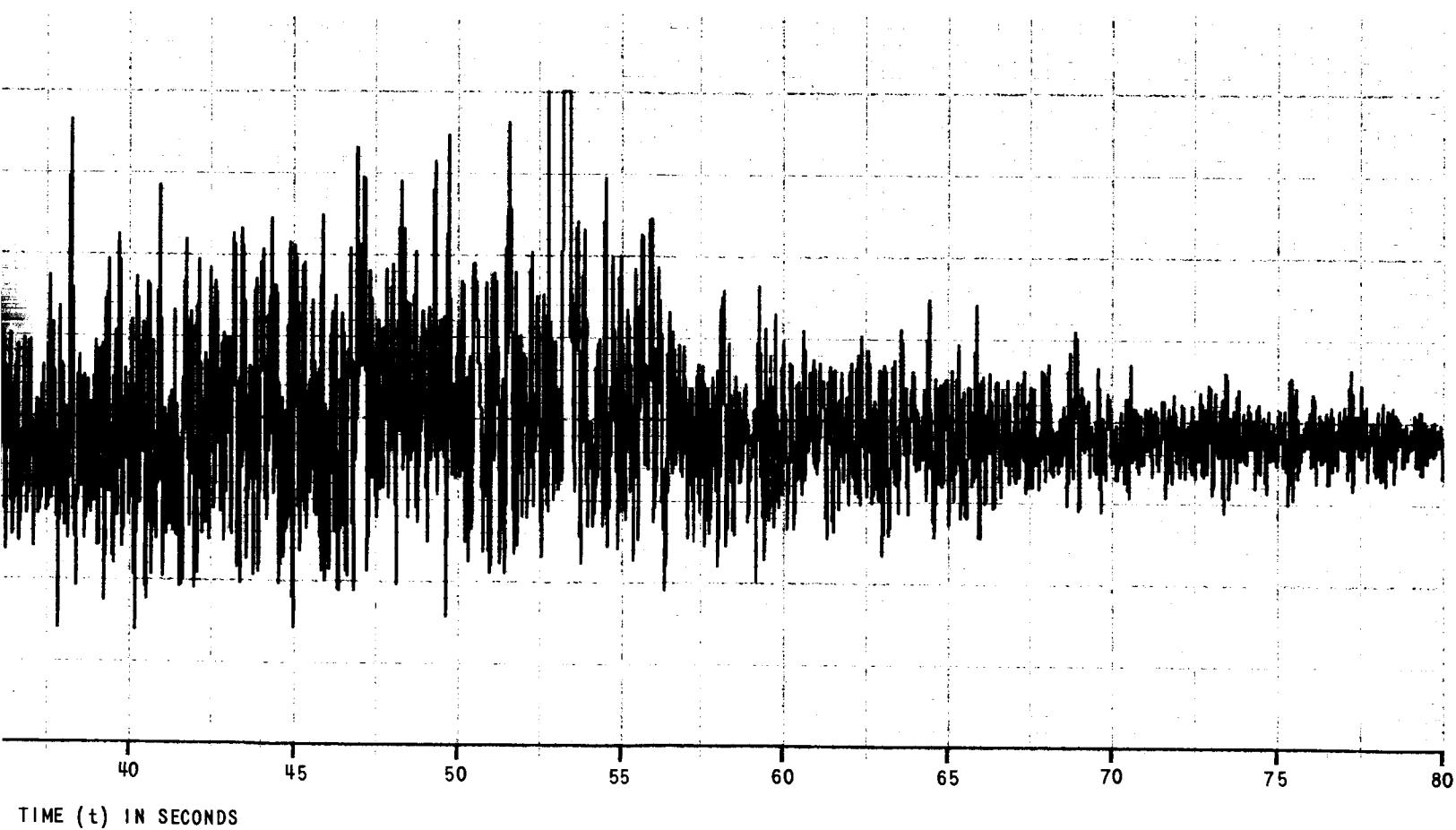


Figure 5 PLOT OF FLEXIBLE BOOSTER TEST VIBRATION  
SIGNAL AS A FUNCTION OF TIME.



Figure 6 shows the correlator output for positive  $\tau$ . The running time scale corresponds to the waveform time scale in Figure 5. The correlator output shows fluctuations in amplitude as a function of both independent variables. Further, for zero delay the amplitude appears to follow the intensity of the waveform itself. Note that certain periodicities become evident from the plot, indicating the bunching of intensity at certain frequencies. Finally, note that for 40 sec.  $< t < 60$  sec., the plot is somewhat raised, indicating a shift in bias. Careful examination of the waveform itself makes this bias shift evident.

A similar analysis was performed for negative values of  $\tau$ . Figure 7 shows a computer plot of the results. Careful examination shows only minor variations between corresponding values of positive and negative  $\tau$ . (Note that the number of computed lines and the two scales are not the same in Figure 6 and 7.)

To more carefully examine the periodicities in the booster data, a spectrum was computed by transforming the correlation plot along the positive  $\tau$  axis. The spectrum was arbitrarily defined as

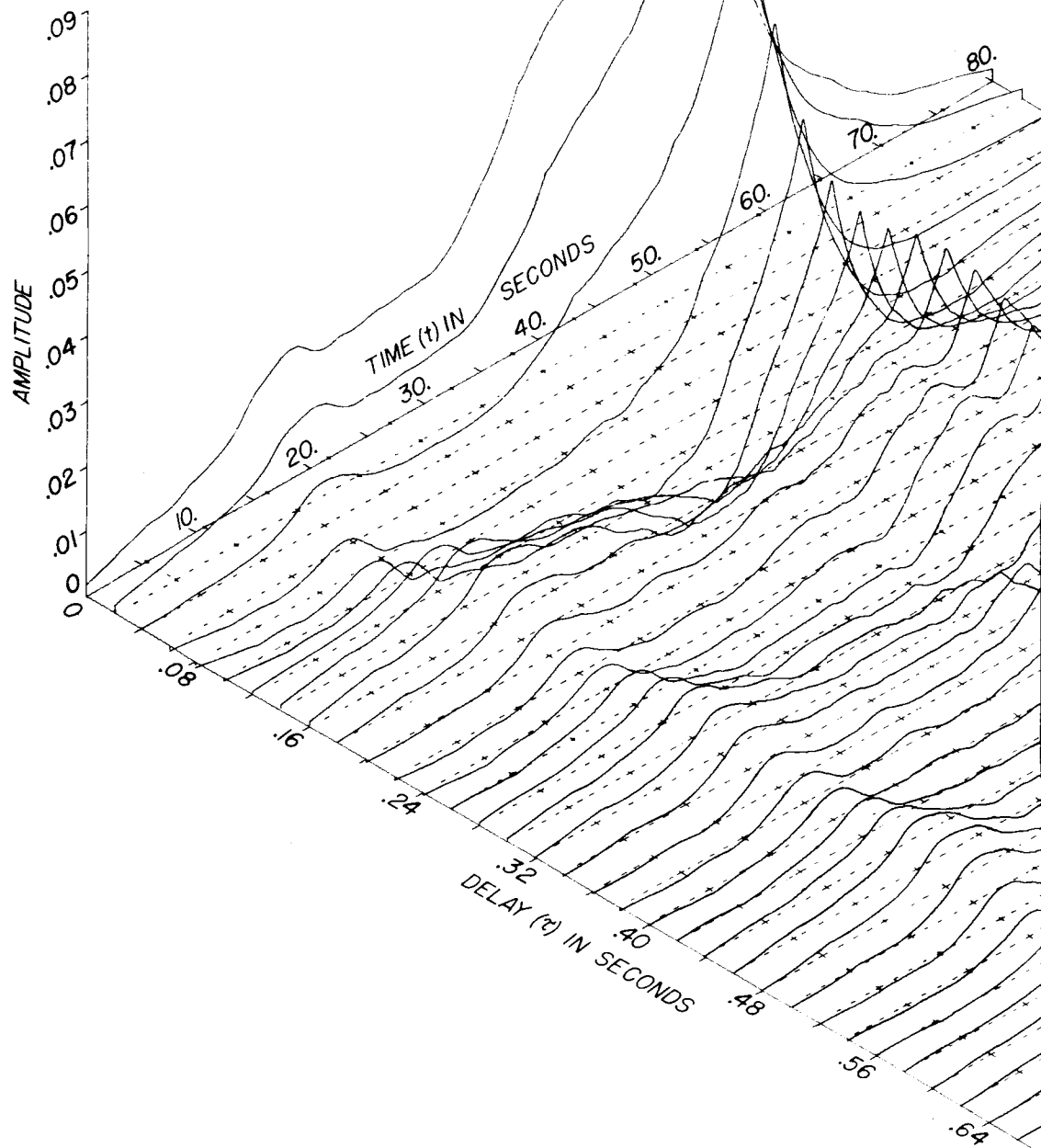
$$\Psi(t, f) \equiv 2 \int_0^{\infty} \psi(t, \tau) \cdot m(\tau) \cdot \cos 2\pi f \tau \cdot d\tau \quad (53)$$

where  $m(\tau)$  is a multiplicative window given for the particular run by

$$m(\tau) = e^{-4.9 \tau^2} \quad (54)$$

Figure 8 is the resulting spectral plot for the booster data. Running time appears as the lower independent variable axis and frequency appears on the upper independent variable axis. Again, running time on the plot corresponds to the waveform time scale of Figure 5.

Examination of the spectral plot shows that a predominance of power lies below 2 Hz and about the frequency of 7 Hz. The plot clearly shows the change in bias, discussed earlier, by its shift in zero frequency



30-1

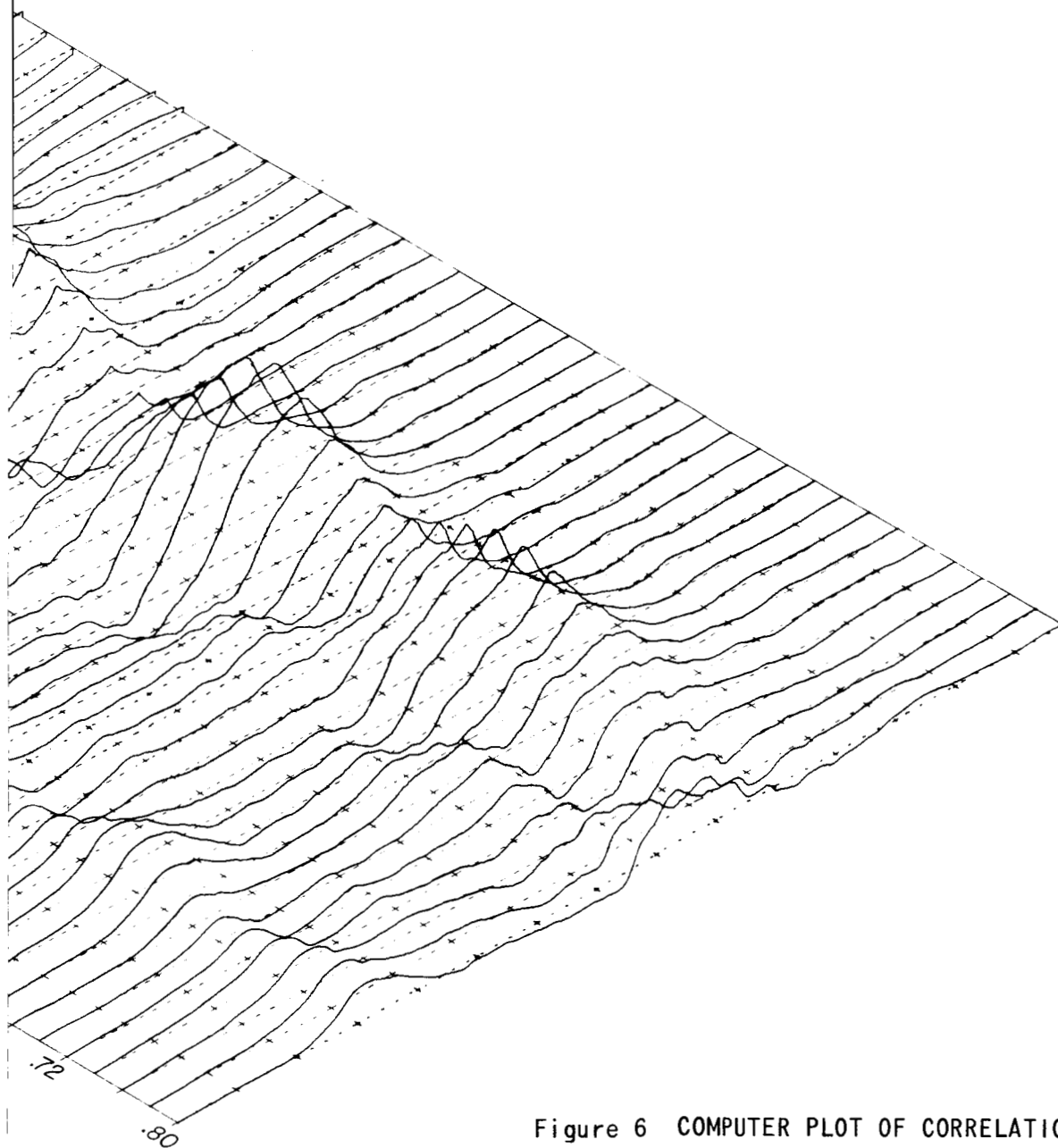


Figure 6 COMPUTER PLOT OF CORRELATION ANALYZER  
OUTPUT FOR AUTOCORRELATION OF BOOSTER  
VIBRATION SIGNAL (POSITIVE  $\tau$  ONLY).

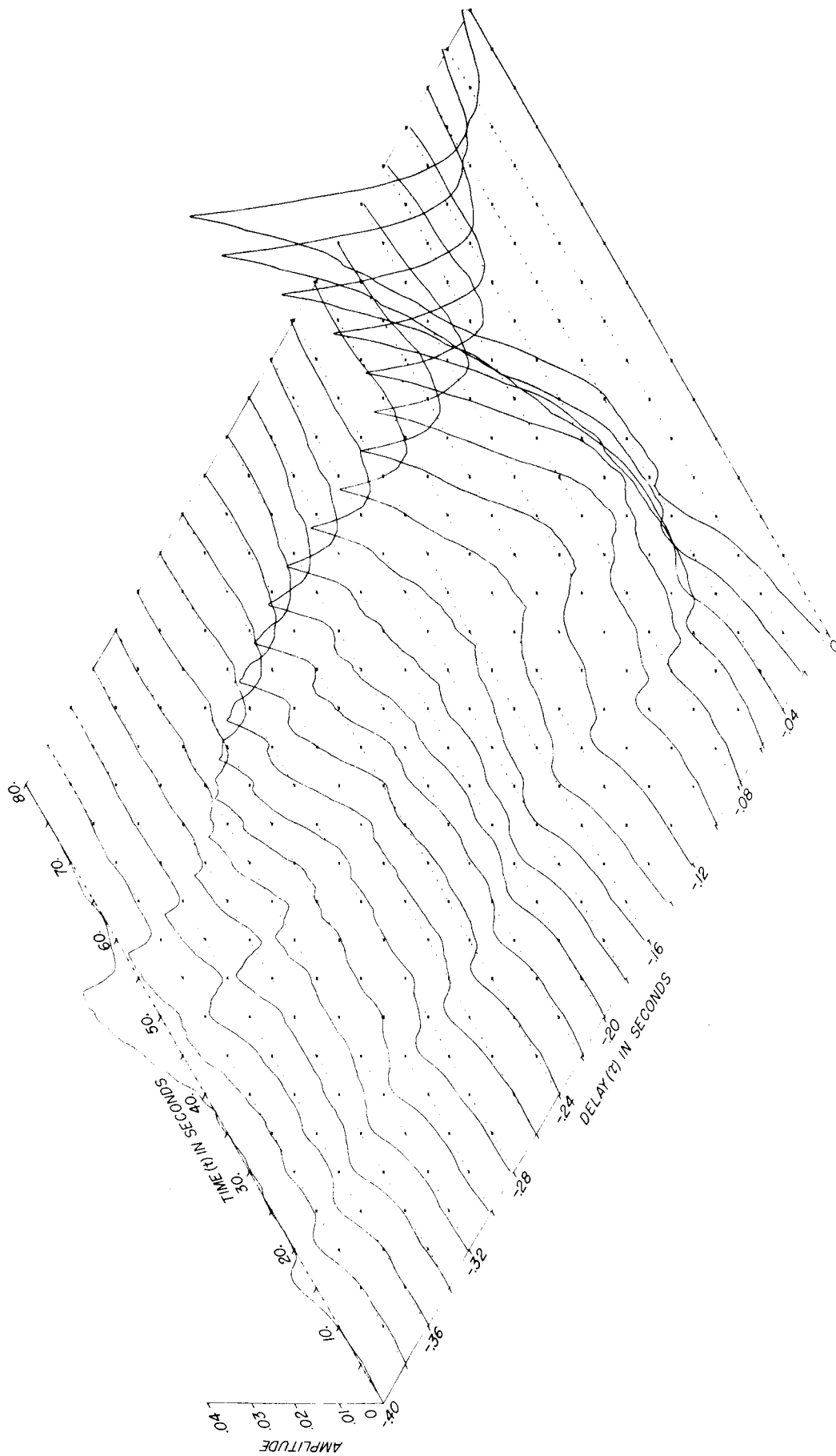
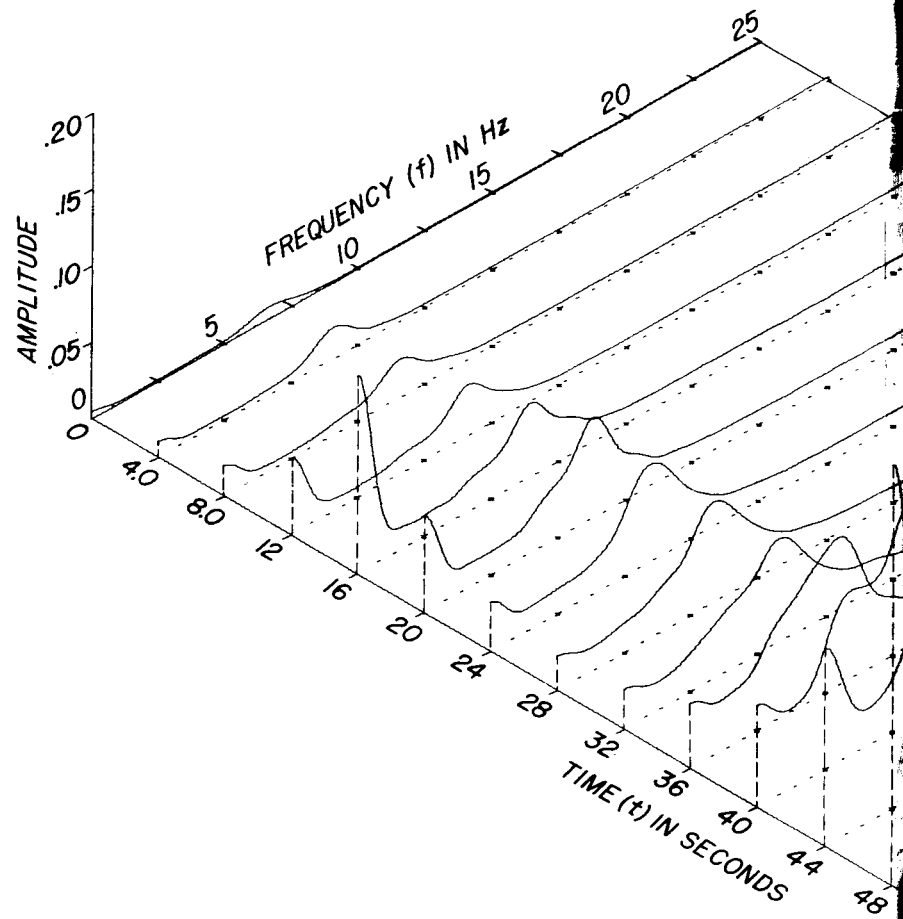


Figure 7 COMPUTER PLOT OF CORRELATION ANALYZER OUTPUT FOR  
AUTOCORRELATION OF BOOSTER VIBRATION SIGNAL (NEGATIVE  $\tau$  ONLY).



32-1

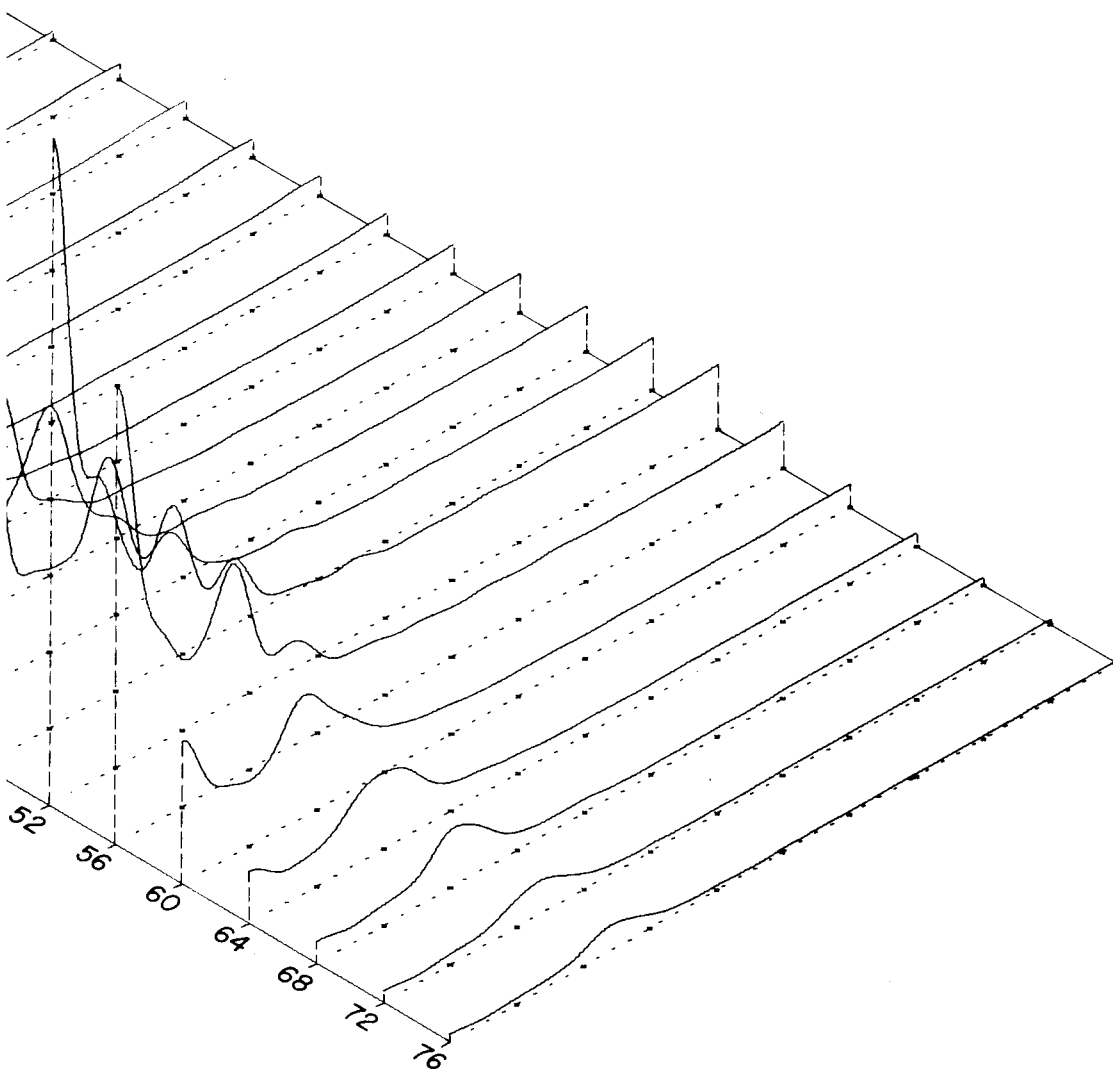


Figure 8 COMPUTER SPECTRAL PLOT OF BOOSTER  
VIBRATION SIGNAL OBTAINED BY  
TRANSFORMING CORRELATION ANALYZER  
PLOT.

32-2

value for 40 sec.  $< t < 60$  sec. Although the amplitude of the power around 7 Hz does fluctuate, the frequency remains relatively constant and is probably attributable to a bending mode. Note that there are no traces of higher frequency resonances, even though these would have been detectable up to 30 Hz.

From this application to booster data it becomes clear that the nonreal-time nonstationary correlation theory, when implemented, is capable of exhibiting important statistical information in practical circumstances. Statistical fluctuations as a function time in nonstationary data can be much more carefully studied with this nonstationary theory.

#### IV. NONREAL-TIME (OFF-LINE) SPECTRUM ANALYSIS OF NONSTATIONARY SIGNALS

##### A. BACKGROUND

Recently, a real-time theory for correlation analysis of nonstationary signals was presented.<sup>2</sup> The theory is based on optimal approximation of the ensemble average real-time correlation function. The nonstationary correlation theory was subsequently extended to the nonreal-time case. (See Section II of this report.) This nonreal-time theory has the advantages that 1) the correlation function may be computed for positive and negative shifts between signal pairs, 2) more general test functions may be used in optimizing the correlation analyzer configuration, and 3) the digital computer can be used for performing the computations.

In some cases, the ultimate desired result is the best estimate of the nonstationary correlation function of the signal being analyzed. However, in other cases the ultimate desired result is the best estimate of the nonstationary spectrum. It is very important to realize that an optimal estimate of the nonstationary correlation function, when Fourier transformed, does not necessarily yield an optimal estimate of the nonstationary spectrum. Although transforming the correlation function to obtain a spectrum may produce satisfactory results, it would be better to obtain the best estimate of the spectrum directly. One could then be assured that the best possible use has been made of the limited data available for nonstationary spectrum analysis.

In view of the problem associated with transforming an optimal correlation function estimate, it was decided to attempt the development of a direct nonreal-time theory for nonstationary spectrum analysis. It was hoped that a companion spectral theory could be developed using the same philosophical approach as is used in the nonreal-time correlation theory. If such a companion theory could be developed, then an investigator could choose the theory which would make his ultimate desired result (i. e., either the spectral or the correlation function) most accurate. This report presents a direct theory of nonreal-time nonstationary spectrum analysis. The objective will be to present the theory, but not to go into great detail regarding



its application. This theory has not yet been experimentally verified or applied. The notation used in this section is similar to that used for the nonstationary correlation theory. In addition, the theory is described in the same framework insofar as possible.

## B. DEFINITIONS AND PROBLEM STATEMENT

Let  $\{ \pi i_1(t) ; \pi i_2(t) \} ; \pi = 1, 2, \dots, P$  represent a sequence of pairs of real signals that have been generated by the same nonstationary random process. The independent variable  $t$  usually represents running time, but can also represent a distance measure in some cases. This sequence represents the given information about the process from which the spectral density is to be estimated.

The crosscorrelation function, whether the process is stationary or nonstationary, is defined as the expected value of the lagged product of  $\pi i_1(t)$  and  $\pi i_2(t)$ :

$$\phi_{12}(t, \tau) \equiv E [\pi i_1(t) \pi i_2(t - \tau)]^* \quad (55)$$

where  $\pi$  is an arbitrary integer and  $\tau$  represents a time translation between the two signals. Beginning with this expected value definition it is possible to show, using the law of large numbers, that in most cases  $\phi_{12}(t, \tau)$  would be equal to the limiting sample mean of lagged signal product pairs. Thus,

$$\phi_{12}(t, \tau) = \lim_{N \rightarrow \infty} \sum_{\pi=1}^N \frac{\pi i_1(t) \pi i_2(t - \tau)}{N} \quad (56)$$

when the right hand side of this equation exists.

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\* In this definition the mean values of  $\pi i_1(t)$  and  $\pi i_2(t)$  are not removed. For a zero mean process, this definition corresponds to the conventional definition.

Define  $\theta_{12}(t, \tau)$  as the sample mean of the given signal lagged products:

$$\theta_{12}(t, \tau) \equiv \sum_{n=1}^P \frac{x_{12}^{(n)}(t) x_{12}^{(n)}(t-\tau)}{P} \quad (57)$$

where again  $\tau$  represents the delay or displacement variable. Further, define a noise-like difference as

$$\eta_{12}(t, \tau) \equiv \theta_{12}(t, \tau) - \phi_{12}(t, \tau) \quad (58)$$

where  $\eta_{12}(t, \tau)$  is assumed extraneous to the correlation function,  $\phi_{12}(t, \tau)$ .

In regard to spectra, define the spectral density of the process as

$$\Phi_{12}(t, \omega) \equiv \int \phi_{12}(t, \tau) e^{-j\omega\tau} d\tau^* \quad (59)$$

and similarly define  $\Theta_{12}(t, \omega)$  as

$$\Theta_{12}(t, \omega) \equiv \int \theta_{12}(t, \tau) e^{-j\omega\tau} d\tau \quad (60)$$

Then if

$$N_{12}(t, \omega) \equiv \int \eta_{12}(t, \tau) e^{-j\omega\tau} d\tau \quad (61)$$

it follows that

$$N_{12}(t, \omega) = \Theta_{12}(t, \omega) - \Phi_{12}(t, \omega) \quad (62)$$

---

\* When limits on integrals are deleted, they are to be taken as  $\infty$  at the upper limits and  $-\infty$  at the lower limits. Note also that in this spectral theory the substitution  $S = j\omega$  has been used, to improve intuitive insight in dealing with spectral densities.

With this group of definitions it becomes possible to postulate the problem of nonreal-time nonstationary spectrum analysis. Suppose that the  $P$  pairs of signals discussed above are given and that by processing these  $P$  pairs, the best estimate of spectral density (59) is to be obtained. When so stated, the problem of nonstationary spectral analysis may be treated analytically.

It is necessary to specify a class of admissible operations on the first  $P$  signal pairs that may be used in approximating  $\Phi_{12}(t, \omega)$ . The class should be chosen so that accurate approximation is possible, yet it should not be chosen so general as to make the computations or optimization process prohibitively difficult. Let the output of the spectral approximation operation be defined as  $X_{12}(t, \omega)$ . Then choose the class of operations such that they may be described as two one-dimensional filtering operations on  $\Theta_{12}(t, \omega)$ ; that is, let

$$X_{12}(t, \omega) = \frac{1}{2\pi} \iint k(\lambda) G(\omega_1) \Theta_{12}(t - \lambda, \omega - \omega_1) d\lambda d\omega_1, \quad (63)$$

where  $k(\lambda)$  is the impulse response function of one filtering operation and  $G(\omega_1)$  is the transfer function of another filtering operation. The two functions are to be specified in the subsequent optimization process in such a way that some measure of the difference between  $\Phi_{12}(t, \omega)$  and  $X_{12}(t, \omega)$  is minimized.

Equation 63 describing the class of admissible operations on the given  $P$  member signal pairs has been chosen to take into account a number of factors that are important in nonstationary spectrum analysis. First, it must be recognized that  $\Phi_{12}(t, \omega)$  and  $\Theta_{12}(t, \omega)$  may only be computed if  $\phi_{12}(t, \tau)$  and  $\theta_{12}(t, \tau)$  are available for all values of  $\tau$ . In practice, it will generally be impossible to compute  $\theta_{12}(t, \tau)$  for all values of  $\tau$ , and therefore, a theory of nonstationary spectrum analysis should take finite record length into account. Note that the spectral theory must take record length into account in a much more direct way than the correlation theory.

The problem of finite record length is accounted for by the operation of Equation 63 . If the inverse transform (with respect to  $\omega$ ) of Equation 63 is taken, the resulting equation is

$$x_{12}(t, \tau) = \int k(\lambda) \theta_{12}(t - \lambda, \tau) g(\tau) d\lambda^* \quad (64)$$

This equation shows clearly that  $g(\tau)$  weights  $\theta_{12}(t, \tau)$  along the  $\tau$  axis; that is,  $g(\tau)$  forms a multiplicative window. If this window,  $g(\tau)$  has negligible value for suitably large magnitudes of  $\tau$ , then  $\theta_{12}(t, \tau)$  need not be computed. It is in this way that finite record length will be taken into account.

The admissible class described by Equation 63 also has the advantage that the computation can be carried out in different ways. These various computational methods will be discussed after the optimization procedure is described.

Yet another advantage of the filtering operation of Equation 63 is that spatial filtering of square data arrays is not required. Because of the high cost of spatial filtering (assuming optical computing techniques are not used), the filtering operation has been limited to two one-dimensional operations.

### C. DEVELOPMENT OF A PERFORMANCE MEASURE

The problem of nonstationary nonreal-time spectrum analysis as postulated in this report may now be stated as the determination of  $k(\lambda)$  and  $G(\omega)$  in such a way as to make the output of the spectral operation  $X_{12}(t, \omega)$  approximate as closely as possible the spectral density  $\Phi_{12}(t, \omega)$ . To perform this optimal approximation, it is necessary to carefully quantify the errors

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\*In this report, functions of  $\omega$  written with upper case letters are Fourier transforms of their corresponding lower case counterparts. For example,

$$G(\omega) = \int g(\tau) e^{-j\omega\tau} d\tau$$

produced by the spectral operation or analyzer and then to minimize these. There are three sources of error in spectrum analysis of nonstationary signals just as there are three in the corresponding correlation analysis problem. However, in addition there is the finite record length constraint that must be incorporated in the spectral analysis case.

The sources of error are readily identified by substituting (62) into (63):

$$X_{12}(t, \omega) = \frac{1}{2\pi} \iint k(\lambda) G(\omega_1) \Phi_{12}(t-\lambda, \omega-\omega_1) d\lambda d\omega_1 \\ + \frac{1}{2\pi} \iint k(\lambda) G(\omega_1) N_{12}(t-\lambda, \omega-\omega_1) d\lambda d\omega_1 \quad (65)$$

The first term on the right shows that the filtering operation produces distortion of the desired spectral density,  $\Phi_{12}(t, \omega)$ , along both axes (independent variables). Examination of the second term on the right shows that the entire quantity must be considered as error, since  $N_{12}(t, \omega)$  is an extraneous or unwanted component of the function  $\Phi_{12}(t, \omega)$ . Therefore, the three sources of error may be classified arbitrarily as:

1. distortion of the spectral density as a function of  $t$ ,
2. distortion of the spectral density as a function of  $\omega$ ,  
and
3. noise or instability resulting from the extraneous component.

In addition to the three sources of error, finite record length must be taken into account.

#### Measures of Distortion in $t$ and $\omega$

In order that the distortion of the spectral density in  $t$  may be minimized, a test-type spectral density function is chosen. This test function should be representative (in the  $t$  dimension) of those being detected. Of course, precise knowledge of the spectral density has been postulated as unavailable, since that type of knowledge would preclude the

need for performing a spectrum analysis. Test function testing of the spectral operation or analyzer affords a good, workable compromise between total absence and total presence of knowledge regarding the process.

Let the test function be given by

$$\Phi_{12}(t, \omega) = \lambda_t q(t) \quad \text{for all } \omega \quad (66)$$

where  $q(t)$  is the test function in  $t$ , and where  $\lambda_t$  is an arbitrary positive constant determining the weighting of the test function. The function  $q(t)$  is made uniform in  $\omega$  in order that separation of distortion in  $t$  and distortion in  $\omega$  may be accomplished.

The test function  $q(t)$  will be restricted in order that steady-state spectrum analyzer outputs possess correct values. In other words, only test functions which adequately test the steady-state output errors will be considered admissible. Thus,  $q(t)$  must be chosen from the class of functions which have the following property:

$$q(t) = \int_{-\infty}^t q_1(\sigma) d\sigma \quad (67)$$

where  $Q_1(\omega)$  is such that  $Q_1(0) \neq 0$ . Equation 67 may be written in the form

$$q(t) = \int q_1(\sigma) u_-(t-\sigma) d\sigma \quad (68)$$

where  $u_-(t)$  is the unit step function. Then the Fourier transform of (68) yields

$$Q(\omega) = \frac{Q_1(\omega)}{j\omega} \quad (69)$$

an equation which will be used later.

A measure of the distortion in  $t$  for the chosen test function is easily obtained by squaring and integrating the difference between the

output of the spectral analyzer (or operation) and the test function itself. Let the measure of distortion be defined as

$$P_t = \lambda_t^2 \int \left[ q(t) - \frac{1}{2\pi} \int G(\omega_1) d\omega_1 \cdot \int k(\lambda) q(t-\lambda) d\lambda \right]^2 dt \quad (70)$$

It will be assumed that  $q(\tau)$  is real, and therefore  $q(0) = \frac{1}{2\pi} \int G(\omega_1) d\omega_1$  is real. Accordingly  $P_t$  is real. Equation 70 will serve to assess the first source of error in nonstationary spectrum analysis.

Distortion in  $\omega$  produced by the spectral analyzer may be handled in a similar manner; however, a frequency domain test function must be used. For testing, let

$$\Phi_{12}(t, \omega) = \lambda_\omega Y(\omega) \quad \text{for all } t \quad (71)$$

where  $Y(\omega)$  is the  $\omega$ -axis test function and  $\lambda_\omega$  is an arbitrary positive weighting constant. The test function is made uniform in  $t$  (equivalent to being stationary), in order that separation of this source of distortion may be obtained. Postulate that  $Y(\omega)$  need not be real, but that its inverse Fourier transform is real.

Again, the test function class is restricted to insure correct steady-state reading of the output of the analyzer. Let  $y(\tau)$ , the inverse Fourier transform of  $Y(\omega)$ , possess the property

$$y(\tau) = O\left(\frac{1}{\tau}\right) \quad \text{as } \tau \rightarrow 0 \quad (72)$$

that is,  $y(\tau)$  is the order of  $\frac{1}{\tau}$  as  $\tau$  approaches zero.

The measure of distortion in  $\omega$  is then defined as

$$P_\omega = \frac{1}{2\pi} \int |e_\omega(\omega)|^2 d\omega \quad (73)$$

where

$$e_\omega(\omega) = \lambda_\omega \left[ Y(\omega) - \int k(\lambda) d\lambda \cdot \frac{1}{2\pi} \int G(\omega_1) Y(\omega - \omega_1) d\omega_1 \right] \quad (74)$$

Note that since  $Y(\omega)$  need not be real that the square of the modulus is used in the measure. If (74) is substituted into (73) and Parseval's theorem is applied, it is found that

$$P_{\omega} = \lambda_{\omega}^2 \int \left[ y(\tau) - \int k(\lambda) d\lambda \cdot y(\tau) \cdot g(\tau) \right]^2 d\tau \quad (75)$$

$$P_{\omega} = \lambda_{\omega}^2 \int y^2(\tau) \left[ 1 - \int k(\lambda) d\lambda \cdot g(\tau) \right]^2 d\tau \quad (76)$$

This relatively simple measure will serve to assess the second form of error in nonstationary spectrum analysis.

#### Error Measure for the Extraneous Component

An extraneous or noise-like component will be present in the output of the spectrum analyzer. This component is represented mathematically by the second term on the right side of (65). This term must be made as small as possible because its entire contribution is extraneous by definition of  $N_{12}(t, \omega)$  in (61). A test function and measure must be selected to properly assess this third source of error.

Consider that the function  $\Phi_{12}(t, \omega)$  and the function  $\Theta(t, \omega)$  can be visualized as two-dimensional surfaces or arrays. The amount by which they differ is again a two-dimensional surface,  $N_{12}(t, \omega)$ . Thus,  $N_{12}(t, \omega)$  can be considered as a two-dimensional noise waveform.

Suppose the noise test function  $N_{12}(t, \omega)$  is chosen to be of infinite length in  $t$ . Then choose the measure

$$P_n = \frac{1}{2\pi} \int \overline{|E_n(t, \omega)|^2} d\omega \quad (77)$$

where

$$E_n(t, \omega) = \frac{1}{2\pi} \iint k(\lambda) G(\omega_1) N_{12}(t - \lambda, \omega - \omega_1) d\lambda d\omega_1 \quad (78)$$

and where the bar indicates a time ( $t$ ) average. Since  $N_{12}(t, \omega)$  will generally be complex, the square of the modulus is used in the measure (77). Note



that an integration of error over frequency and an average of error in time is incorporated in the measure. Since the extraneous component is a "mixed" function (with time and frequency variables), it is not surprising that a different type of summing operation must be used for each axis of the measure.

By an extended form of Parseval's theorem, the performance measure may be written as

$$P_n = \int e_n^2(t, \tau) d\tau \quad (79)$$

where  $e_n(t, \tau)$ , the inverse Fourier transform of  $E_n(t, \omega)$ , is given by

$$e_n(t, \tau) = \int k(\lambda) n_{12}(t - \lambda, \tau) g(\tau) d\lambda \quad (80)$$

Evaluating  $P_n$  then yields

$$P_n = \iiint k(\lambda_1) k(\lambda_2) \overline{n_{12}(t - \lambda_1, \tau) n_{12}(t - \lambda_2, \tau)} g^2(\tau) d\tau d\lambda_1 d\lambda_2 \quad (81)$$

At this point,  $N_{12}(t, \omega)$  is further specified and limited by placing restrictions on  $n_{12}(t, \tau)$ . Let  $n_{12}(t, \tau)$  possess the property

$$\overline{n_{12}(t - \lambda_1, \tau) n_{12}(t - \lambda_2, \tau)} = c_n(\lambda_2 - \lambda_1) \cdot f_n(\tau) \quad (82)$$

The function  $f_n(\tau)$  may be rapidly fluctuating, since it is not the result of any smoothing operation. The function  $c_n(\lambda)$  is seen to be even in  $\lambda$  by examination of Equation 82 for any fixed value of  $\tau$ . Then, substitution of (82) into (81) yields the final form of the measure for assessing the noise or instability in nonstationary spectrum analysis:

$$P_n = \iint k(\lambda_1) k(\lambda_2) c_n(\lambda_2 - \lambda_1) d\lambda_1 d\lambda_2 \cdot \int f_n(\tau) g^2(\tau) d\tau \quad (83)$$

### Finite Record Length Constraint

In addition to the three sources of error described above, account must be taken of the problem of finite record length. Whereas the problem of record length indirectly enters into nonstationary correlation analysis, it enters directly into nonstationary spectrum analysis. Examination of the definitions of  $\bar{\Phi}_{12}(t, \omega)$  (59) and  $\bar{\Theta}_{12}(t, \omega)$  (60) shows that all values of  $\tau$  are required for the computation of the spectrum at any frequency  $\omega$ . The spectral operation or analyzer configuration has been selected to account for finite record length. Equation 64 shows that  $g(\tau)$  forms a multiplicative window, thereby limiting the required length of data (at the cost of loss of frequency domain resolution).

The constraint on record length will be incorporated by obtaining a measure which increases as  $g(\tau)$  takes on greater spread on the  $\tau$  axis. By penalizing for large width in  $\tau$ , record length may be held to a minimum. Let

$$P_b = \int v(\tau) g^2(\tau) d\tau \quad (84)$$

where  $P_b$  is the measure of spread, and  $v(\tau)$  is an arbitrary positive weighting function which may be chosen to weight heavily the contribution to  $P_b$  occurring at large values of  $\tau$ .

### The Total Performance Measure

A total assessment of the errors involved in nonreal-time nonstationary spectrum analysis can be obtained by summing the performance measures for the three sources of error and the record length penalty measure. By changing  $\lambda_t$  and  $\lambda_\omega$ , as well as the test functions, the various sources of error may be traded against one another. Let the total measure be defined as

$$P_T \equiv P_t + P_\omega + P_n + P_b \quad (85)$$

In addition to this measure, two constraints must be chosen which force the spectrum analyzer to properly indicate steady-state conditions. These conditions insure that, even though distortion occurs on each axis of the spectral density when detected, the distortion does not shift the average level of the spectral density. The constraints are

$$\frac{1}{2\pi} \int G(\omega) d\omega = g(0) = 1.0 \quad (86)$$

and

$$\int k(\lambda) d\lambda = K(0) = 1.0 \quad (87)$$

These constraints will be incorporated by introducing Lagrange multiplier side conditions into the equations to be extremized.

As in the correlation theory, this spectrum analysis theory has been developed in terms of performance measures for assessing error. The approach and specific test function forms chosen may appear largely arbitrary. It is admitted here that a number of different approaches might be taken. The one presented here has been selected because it is relatively general but manageable, it leads to a precise optimal solution (in the sense of the performance measure), and it properly reflects the sources of error and desired constraints.

#### D. DETERMINATION OF THE OPTIMUM SPECTRUM ANALYZER

The minimization of the performance measure is accomplished by determining the extremals of the following equation:

$$\begin{aligned} I = & \lambda_t^2 \int \left[ q(t) - g(0) \cdot \int k(\lambda) q(t-\lambda) d\lambda \right]^2 dt + \lambda_\omega^2 \int y^2(\tau) \left[ 1 - \int k(\lambda) d\lambda \cdot g(t) \right]^2 d\tau \\ & + \iint k(\lambda_1) k(\lambda_2) c_n(\lambda_2 - \lambda_1) d\lambda_1 d\lambda_2 \cdot \int f_n(\tau) g^2(\tau) d\tau \\ & + \int v(\tau) g^2(\tau) d\tau + \lambda_a [g(0) - 1] + \lambda_b \left[ \int k(\lambda) d\lambda - 1 \right] \end{aligned} \quad (88)$$

where  $\lambda_a$  and  $\lambda_b$  are Lagrange multipliers to be determined. Two different extremal equations must be obtained, one in which  $k(\lambda)$  is subjected to a variation and one in which  $G(\omega)$  is subjected to a variation. The objective is to determine the functions  $k(\lambda)$  and  $G(\omega)$  in such a way that the performance measure  $P_T$  is minimized and the two steady-state constraint conditions are satisfied.

The first extremal equation, with  $g(\tau)$  subjected to a variation, is obtained by the usual calculus-of-variations approach. The function  $g(\tau)$  is replaced by  $g(\tau) + \epsilon_g \eta_g(\tau)$  where  $\eta_g(\tau)$  is an arbitrary differentiable function and  $\epsilon_g$  is an arbitrary small parameter. Substitution into (87) yields the equation

$$\begin{aligned}
 I_g = & \lambda_t^2 \int \left\{ \dot{q}(t) - [g(0) + \epsilon_g \eta_g(0)] \cdot \int k(\lambda) \dot{q}(t - \lambda) d\lambda \right\}^2 dt \\
 & + \lambda_\omega^2 \int y^2(\tau) \left\{ 1 - \int k(\lambda) d\lambda \cdot [g(\tau) + \epsilon_g \eta_g(\tau)] \right\}^2 d\tau \\
 & + \int \int k(\lambda_1) k(\lambda_2) c_n(\lambda_2 - \lambda_1) d\lambda_1 d\lambda_2 \cdot \int f_n(\tau) [g(\tau) + \epsilon_g \eta_g(\tau)]^2 d\tau \\
 & + \int v(\tau) [g(\tau) + \epsilon_g \eta_g(\tau)]^2 d\tau + \lambda_a [g(0) + \epsilon_g \eta_g(0) - 1] \\
 & + \lambda_b \left[ \int k(\lambda) d\lambda - 1 \right]
 \end{aligned} \tag{89}$$

Then the following equation is formed

$$\left. \frac{\partial I_g}{\partial \epsilon_g} \right|_{\epsilon_g=0} = 0 \tag{90}$$

yielding

$$\begin{aligned}
& \lambda_t^2 \eta_g(0) \int \left\{ q(t) - g(0) \int k(\lambda) q(t-\lambda) d\lambda \right\} \cdot \left\{ \int k(\lambda) q(t-\lambda) d\lambda \right\} dt \\
& + \lambda_\omega^2 \int y^2(\tau) \left\{ 1 - \int k(\lambda) d\lambda \cdot g(\tau) \right\} \cdot \left\{ \int k(\lambda) d\lambda \cdot \eta_g(\tau) \right\} d\tau \\
& - \int v(\tau) g(\tau) \eta_g(\tau) d\tau - \frac{1}{2} \lambda_a \eta_g(0) - K_2 \int f_n(\tau) g(\tau) \eta_g(\tau) d\tau = 0
\end{aligned} \tag{91}$$

where

$$K_2 \equiv \iint k(\lambda_1) k(\lambda_2) c_n(\lambda_2 - \lambda_1) d\lambda_1 d\lambda_2 \tag{92}$$

The parameter  $\lambda_a$  in the above extremal equation must assume a value that allows the constraint on  $g(\tau)$ , Equation 86, to be satisfied. Suppose that  $\lambda_a$  is chosen to satisfy the following equation:

$$-\frac{1}{2} \lambda_a \eta_g(0) + \lambda_t^2 \eta_g(0) \int \left\{ q(t) - g(0) \int k(\lambda) q(t-\lambda) d\lambda \right\} \cdot \left\{ \int k(\lambda) q(t-\lambda) d\lambda \right\} dt = 0 \tag{93}$$

Then, if for this value of  $\lambda_a$ , the resulting extremal equation satisfies the constraint, a solution will have been obtained. Substitution of (93) into (91) and subsequent application of the fundamental theorem of the calculus of variations yields an extremal equation of the form

$$\begin{aligned}
& \lambda_\omega^2 y^2(\tau) \left[ 1 - \int k(\lambda) d\lambda \cdot g(\tau) \right] \int k(\lambda) d\lambda \\
& - K_2 f_n(\tau) g(\tau) - v(\tau) g(\tau) = 0
\end{aligned} \tag{94}$$

which may be solved for  $g(\tau)$  :

$$g(\tau) = \frac{\lambda_\omega^2 y^2(\tau) K(0)}{[K_2 f_n(\tau) + v(\tau)] + \lambda_\omega^2 y^2(\tau) K^2(0)} \quad (95)$$

where  $K(0)$  is defined in (87). Equation 95 will be shown later to satisfy the constraint relation.

The second extremal equation, with  $k(\lambda)$  subjected to a variation, is obtained by the same mathematical method. The major equations become

$$\begin{aligned} I_k = & \lambda_t^2 \int \left\{ q(t) - g(0) \int [k(\lambda) + \mathcal{E}_k \eta_k(\lambda)] q(t-\lambda) d\lambda \right\}^2 dt \\ & + \lambda_\omega^2 \int y^2(\tau) \left\{ 1 - \int [k(\lambda) + \mathcal{E}_k \eta_k(\lambda)] d\lambda \cdot g(\tau) \right\}^2 d\tau \\ & + \iint [k(\lambda_1) + \mathcal{E}_k \eta_k(\lambda_1)] [k(\lambda_2) + \mathcal{E}_k \eta_k(\lambda_2)] c_n(\lambda_2 - \lambda_1) d\lambda_1 d\lambda_2 \cdot G_2 \\ & + \int v(\tau) g^2(\tau) d\tau + \lambda_a [g(0) - 1] + \lambda_b \left\{ \int [k(\lambda) + \mathcal{E}_k \eta_k(\lambda)] d\lambda - 1 \right\} \end{aligned} \quad (96)$$

where

$$G_2 = \int f_n(\tau) g^2(\tau) d\tau \quad (97)$$

$$\left. \frac{\partial I_k}{\partial \mathcal{E}_k} \right|_{\mathcal{E}_k=0} = 0 \quad (98)$$

$$\begin{aligned}
& -\lambda_t^2 \int [q(t) - g(0) \int k(\lambda_1) q(t - \lambda_1) d\lambda_1] \left[ g(0) \int \eta_k(\lambda) q(t - \lambda) d\lambda \right] dt \\
& -\lambda_\omega^2 \int y^2(\tau) \left[ 1 - \int k(\lambda_1) d\lambda_1 \cdot g(\tau) \right] \left[ \int \eta_k(\lambda) d\lambda \right] g(\tau) d\tau \\
& + \iint \eta_k(\lambda) k(\lambda_1) c_n(\lambda - \lambda_1) d\lambda_1 d\lambda \cdot G_2 + \frac{1}{2} \lambda_b \int \eta_k(\lambda) d\lambda = 0
\end{aligned} \tag{99}$$

Suppose  $\lambda_b$  is chosen to satisfy the equation

$$\frac{1}{2} \lambda_b - \lambda_\omega^2 \int y^2(\tau) \left[ 1 - \int k(\lambda_1) d\lambda_1 \cdot g(\tau) \right] g(\tau) d\tau = 0 \tag{100}$$

The extremal equation then becomes

$$\begin{aligned}
& -\lambda_t^2 \int [q(t) - g(0) \int k(\lambda_1) q(t - \lambda_1) d\lambda_1] g(0) q(t - \lambda) dt \\
& + G_2 \int k(\lambda_1) c_n(\lambda - \lambda_1) d\lambda_1 = 0
\end{aligned} \tag{101}$$

At this point a somewhat different approach is taken for the second extremal equation. Since Equation 101 is an integral equation with infinite limits on all integrals, it may be Fourier transformed with respect to the variable  $\lambda$ . The resulting equation is

$$-\lambda_t^2 g(0) \left[ \frac{Q_r(\omega)}{j\omega} - g(0) K(\omega) \frac{Q_r(\omega)}{j\omega} \right] \frac{Q(-\omega)}{-j\omega} + G_2 K(\omega) C_n(\omega) = 0 \tag{102}$$

The function  $c_n(\lambda)$  is both real and even in  $\lambda$ ; thus  $C_n(\omega)$  is also real and even and may be written as the following product

$$C_n(\omega) = C_r(\omega) C_r(-\omega) \tag{103}$$

Substitution of (103) into (102) and subsequent solution for  $K(\omega)$  yields

$$K(\omega) = \frac{\left( \frac{\lambda_t^2}{G_2} \right) g(0) \frac{Q_r(\omega)}{j\omega} \cdot \frac{Q_r(-\omega)}{-j\omega}}{C_r(\omega) C_r(-\omega) + \left( \frac{\lambda_t^2}{G_2} \right) g^2(0) \frac{Q_r(\omega)}{j\omega} \cdot \frac{Q_r(-\omega)}{-j\omega}} \tag{104}$$

It must now be shown that  $g(\tau)$  in (95) and  $K(\omega)$  in (104) satisfy the two constraint relations, (86) and (87). Suppose that in (95) and (104),  $\tau$  and  $\omega$  simultaneously approach zero. The two equations

then approach the following values

$$g(\tau) \sim \frac{\lambda_w^2 \frac{b^2}{\tau^2} K(0)}{a + \lambda_w^2 \frac{b^2}{\tau^2} K^2(0)} \quad * \quad (105)$$

$$K(\omega) \sim \frac{\left(\frac{\lambda_t^2}{G_2}\right) g(0) \frac{d}{j\omega} \cdot \frac{d}{-j\omega}}{c + \left(\frac{\lambda_t^2}{G_2}\right) g^2(0) \frac{d}{j\omega} \cdot \frac{d}{-j\omega}} \quad (106)$$

where  $a, b, c,$  and  $d$  are constants. Consequently, in the limit, the two equations becomes

$$g(0) = \frac{K(0)}{K^2(0)} = \frac{1}{K(0)} \quad (107)$$

and

$$K(0) = \frac{g(0)}{g^2(0)} = \frac{1}{g(0)} \quad (108)$$

Thus, the constraint conditions can be satisfied, since (107) and (108) admit the conditions  $g(0) = 1$  and  $K(0) = 1$ . The final expressions for  $g(\tau)$  and  $K(\omega)$  become

$$g(\tau) = \frac{\left(\frac{\lambda_w^2}{K_2}\right) y^2(\tau)}{\left[f_n(\tau) + \frac{v(\tau)}{K_2}\right] + \left(\frac{\lambda_w^2}{K_2}\right) y^2(\tau)} \quad ** \quad (109)$$

---

\* In (105) use has been made of (72).

\*\* Proof that  $g(\tau)$  as given in (109) produces minimum spectral analyzer error (according to the performance measure with constraints incorporated) may be obtained in a straightforward manner. The extremal condition (91) is substituted into the expression  $I_g - P_T$ . Then, the resulting expression is shown to be greater than or equal to zero for all  $\epsilon_g$  and  $\eta_g(\tau)$  which satisfy the constraint condition. A similar proof may be obtained which shows that  $K(\omega)$  as given in (110) produces minimum spectral analyzer error (according to the performance measure with constraints incorporated).



and

$$K(\omega) = \frac{\left(\frac{\lambda_t^2}{G_2}\right) \frac{Q_1(\omega)}{j\omega} \cdot \frac{Q_1(-\omega)}{-j\omega}}{C_1(\omega) C_1(-\omega) + \left(\frac{\lambda_t^2}{G_2}\right) \frac{Q_1(\omega)}{j\omega} \cdot \frac{Q_1(-\omega)}{-j\omega}} \quad (110)$$

It is seen that (109) is a time domain equation, whereas (110) is a frequency domain equation. Again, this domain contrast in the solutions is not surprising in view of the mixed-domain nature of the spectrum analyzer or operation (63). Note that  $K(\omega)$  in (110) is always real and even in  $\omega$ . Therefore, its inverse Fourier transform  $k(\lambda)$  is also real and even in  $\lambda$ . The function  $g(\tau)$  will be even in  $\tau$  if  $f_n(\tau)$ ,  $v(\tau)$  and  $y(\tau)$  are chosen as even functions. However, in general,  $g(\tau)$  need not be even. Finally, note that the quantities  $K_2$  and  $G_2$  appear in (109) and (110) in a way which does not affect the form of solution. Adjustment of  $\lambda_\omega$ ,  $\lambda_t$  and the gain associated with  $v(\tau)$  makes it possible to obtain solutions independent of the values of  $K_2$  and  $G_2$ . Generally, frequency bandwidth and time duration considerations will govern the settings of the parameters  $\frac{\lambda_\omega^2}{K_2}$ ,  $\frac{\lambda_t^2}{G_2}$  and the gain of  $\frac{v(\tau)}{K_2}$ . Thus, their values in relation to the original performance measure  $I$  are unimportant. Equations 109 and 110 complete the specification of the optimal spectral analyzer or operation given in (63).

#### E. METHODS FOR IMPLEMENTING THE OPTIMUM SPECTRUM ANALYZER

As stated earlier, the allowable class of operations for performing the spectrum analysis has been chosen both to allow good spectral estimates and to lend versatility in implementation. In this section, the various methods for realizing the spectrum analyzer will be described in detail.

The first method of computation involves the direct use of Equation 63. In this case, the procedure is 1) computing  $\theta_{12}(t, \tau)$  from (57) for the available data, 2) Fourier transforming (60) yielding  $\Theta_{12}(t, \omega)$  3) filtering the resulting two-dimensional array along in  $t$  by a filter whose impulse response is  $k(\lambda)$ , and 4) convolving the new two-dimensional array

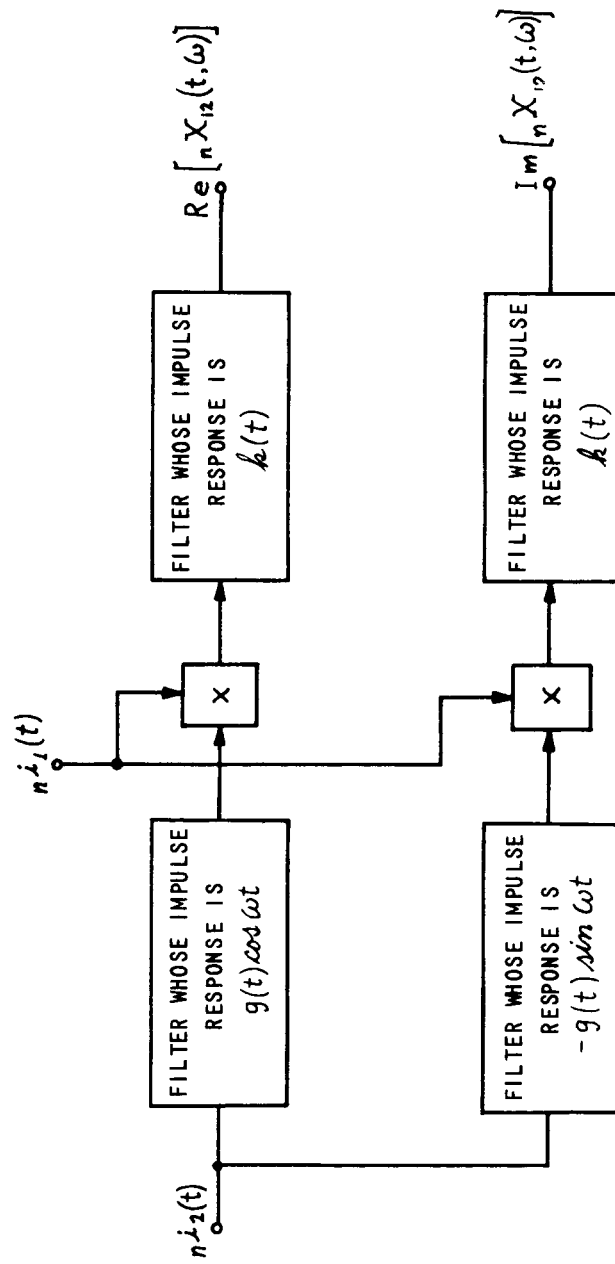


Figure 9 NETWORK CONFIGURATION THAT MAY BE USED FOR IMPLEMENTING THE THIRD APPROACH TO NONREAL-TIME NONSTATIONARY SPECTRAL COMPUTATION.

along the  $\omega$  -axis with the function  $G(\omega)$  . The result will then be  $X_{12}(t, \omega)$  the best estimate of the spectral density.

The second approach to the computation is a variation of the first and makes use of Equation 64 . The steps involve 1) computing  $\theta_{12}(t, \tau)$  from (57) for the available data, 2) multiplying by the "window" function  $g(\tau)$ , 3) filtering the resulting two-dimensional array in the  $t$  -variable by a filter whose impulse response is  $k(\lambda)$ , and 4) Fourier transforming the resulting array with respect to  $\tau$  . Step 3 yields  $X_{12}(t, \tau)$  (Equation 64 ), and step 4 yields  $X_{12}(t, \omega)$  (Equation 63 ).

The third approach allows a direct spectral computation. It is accomplished by dealing individually with each given pair of signals. Figure 9 is a block diagram of the computational configuration. First the signal  $x_{12}(t)$  is passed through two resonant filters; then the outputs of these filters are multiplied by  $x_{12}(t)$ . Each product is then filtered again resulting in real and imaginary parts of the component spectra,  $x_{12}(t, \omega)$  . By repeating the operation for each signal pair up to  $P$  , the optimal spectral estimate is obtained

$$X_{12}(t, \omega) = \sum_{n=1}^P \frac{x_{12}(t, \omega)}{P} \quad (111)$$

Although all three methods are mathematically equivalent and will yield the same answer if properly executed, each one offers certain advantages. The first two methods are probably best suited for computation on a modern full-size digital computer. The second method is particularly attractive for the large digital computer installation because the method is efficient, but requires a good deal of memory. The third method, on the other hand, appears best suited for the hybrid or small digital computer. This method is probably not as efficient as the second, but it also does not require a great deal of memory. The third method allows the value of the spectrum to be read out individually for each setting of  $\omega$  .

## F. CONCLUDING REMARKS IN REGARD TO NONSTATIONARY SPECTRUM ANALYSIS

A number of topics regarding this nonreal-time nonstationary spectrum analysis theory remain to be investigated. Because of the limited scope of the present study, it has been economically impossible to carry this spectrum analysis approach beyond the basic theoretical derivation described in this report. However, as can be seen from the complexity of this theoretical derivation, it represents an important advancement in the field of nonstationary data processing.

Among the topics remaining to be investigated are the following. First, experience must be gained with the choice of the various test functions and parameters; namely,  $\frac{\lambda_{\omega}^2}{K_2}$ ,  $\frac{\lambda_t^2}{G_2}$ ,  $\gamma(\omega)$ ,  $f_n(\tau)$ ,  $v(\tau)$ ,  $g(t)$  and  $c(\lambda)$ . These parameters must be chosen in a way which matches the spectrum analyzer to the data being analyzed. However, the test functions should be chosen with the simplest possible forms in order that complexity in the digital computations may be minimized. Second, numerical examples of the optimal filters (109) and (110) should be computed. Here again, the objective is to gain insight and experience with these filters. Third, the nonstationary spectrum analyzer theory must be experimentally verified. With a derivation as complex as the one described in this report, experimental verification is required as a check on the theory and to make sure that nothing has been overlooked. Finally, the theory should be implemented in a digital program and applied to typical nonstationary data. The objective is to insure that the approach yields meaningful results when applied.

## V. RECOMMENDATIONS

In this section, several areas are recommended for future investigation. They have been chosen because their successful completion will allow the Computation Laboratory's goal of workable programs for nonstationary processing to be realized.

1. Perform an experimental verification of the nonreal-time nonstationary spectral analysis theory.

Because of the limited scope of the present research effort, experimental verification of the new nonstationary spectral analysis theory could not be undertaken. The theory must be verified and refined before it is applied. This step should be considered as essential to the orderly progression of the investigation.

2. Develop a digital computer program for nonreal-time nonstationary spectrum analysis and refine the present digital program for nonreal-time nonstationary correlation analysis.

Either following the experimental verification of the nonstationary spectral analysis theory or concurrent with it, a digital program must be developed for performing spectral analyses. Here again, since the ultimate objective is computation using the digital computer, this step must be considered as essential. Moreover, although a digital program now exists for performing nonreal-time nonstationary correlation analyses, this program should be refined. The original purpose of the program was experimental verification of the correlation theory. Thus, a more general program should be developed.

If sufficient funding is available, then discrete nonstationary correlation and spectral theories for nonreal-time analysis should be developed. Implementation of these discrete theories would make it unnecessary to approximate digitally the continuous filters required in the presently developed theories.

3. Apply the digital programs for nonreal-time nonstationary spectrum and correlation analysis to nonstationary data that the Computation Laboratory is asked to analyze.

The nonstationary theories have been developed so as to be flexible and widely applicable. Accordingly, parameters must be determined experimentally in each application. The intent would be to determine the optimum choice of parameters for the data encountered by the Computation Laboratory and to assure that the programs work satisfactorily. This step should be considered as essential, because it ties the foregoing theoretical results to the required application.

4. Improve the efficiency of the digital programs for nonstationary analysis.

Unlike stationary analysis, nonstationary analysis introduces a second dimension. Consequently, computation times for nonstationary analysis can be expected to be greater than they are for stationary analysis. Methods can be developed which improve the efficiency of nonstationary processing. If the original nonstationary programs are costly to run, it will be important to develop methods for improving their efficiency.

5. Incorporate the digital programs for nonstationary data processing into the Computation Laboratory's program battery.

Certain problems will be encountered in adapting the final digital program to the battery. The plotting equipment used at the Computation Laboratory is different from that used at CAL. Also, a method for interrogating the user of the nonstationary program might be developed. The idea is to have the user supply a few constants regarding his data that will allow "tailoring" of the nonstationary programs to the data. Alternatively, it might be possible to perform a preliminary analysis on the data to allow automatic determination of the analyzer parameters.

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